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Counter-Examples to the Conjecture $M(2k, k+1) = 3k - 1 + \lfloor\frac{k-1}{2}\rfloor$ *

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Abstract: The maximum jump number $M(n,k)$ over a class of $n \times n$ matrices of zeros and ones with constant row and column sum k has been investigated by Brualdi and Jung in [1] where they proposed the conjecture

$$M(2k, k+1) = 3k - 1 + \lfloor\frac{k-1}{2}\rfloor.$$

In this note, we give two counter-examples to this conjecture.

Key words: jump number; (0,1)-matrices; conjecture; counter-examples

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Let $A = [a_{ij}]$ be an m-by-n matrix. Throughout this paper, $s(A)$ and $b(A)$ respectively denote the jump number of P_A and the stair number of P_A as defined in [1]. We use $\Lambda(n,k)$ for the set of all $(0,1)$-matrices of order n with k 1's in each row and column. While $M(n,k) = \max\{s(A) : A \in \Lambda(n,k)\}$. We also use $J_{a,b}$ to denote the a-by-b matrix with all 1's.

The number $M(n,k)$ is defined and studied by R. A. Brualdi and H. C. Jung for the first time in [1], and they also conjectured that

$$M(2k, k+1) = 3k - 1 + \lfloor\frac{k-1}{2}\rfloor$$

holds for every positive integer k. They proved hat $M(2k, k+1) \geq 3k - 1 + \lfloor\frac{k-1}{2}\rfloor$ for all k, and the equality holds for $5 \geq k \geq 1$. Here, we give two examples to disprove this conjecture.

The following two lemmas come from the Theorem 2.2 in [1] and the Corollary 4.11 in [2] respectively .

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Lemma 1 *Let A be a (0,1)-matrix with no zero row or column. Let $b(A) = p$. Then there exist permutation matrices R and S and integers $m_1, \cdots, m_p$ and $n_1, \cdots, n_p$ such that RAS equals*

$$\begin{bmatrix} J_{m_1,n_1} & A_{1,2} & \cdots & A_{1,p} \\ 0 & J_{m_2,n_2} & \cdots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{m_p,n_p} \end{bmatrix}.$$

Lemma 2 *If $k \not| n$ and $n \bmod k \not| k$ for $1 \le k \le n$, then $M(n,k) < 2n - 1 - \lceil \frac{n}{k} \rceil$.*

Example 1 Let

$$A = \begin{bmatrix} J_{4,4} & J_{4,2} & 0 & 0 & J_{4,2} \\ J_{2,4} & J_{2,2} & J_{2,2} & 0 & 0 \\ J_{2,4} & 0 & 0 & J_{2,4} & 0 \\ 0 & J_{2,2} & J_{2,2} & J_{2,4} & 0 \\ 0 & 0 & J_{4,2} & J_{4,4} & J_{4,2} \end{bmatrix},$$

then $A \in \Lambda(14,8)$ and $b(A) = 3$, and it follows that $M(14,8) = 24$.

Proof First we have $b(A) = b(B)$, where

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Obviously, $3 \le b(B) \le 4$. If $b(B) = 4$, then by Lemma 1, there exist two permutation matrices R and S such that

$$RBS = \begin{bmatrix} 1 & 1 & b_{13} & b_{14} & b_{15} \\ 1 & 1 & b_{23} & b_{24} & b_{25} \\ 0 & 1 & b_{33} & b_{34} & b_{35} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where $b_{ij} = 0$ or 1, and each b_{ij} satisfies that the i-th row and the j-th column respectively contain exactly two 0's. We can easily compute $\det(RBS) = 0$, -1 or 1. But $\det(B) = 4$, which contradicts $\det(RBS) = \det(B)$. Hence $b(B) = 3$ or $b(A) = 3$. It follows that $s(A) = 28 - 1 - b(A) = 24$. Therefore, $M(14,8) \ge 24$.

On the other hand, we have $M(14,8) < 28 - 1 - \lceil \frac{14}{8} \rceil = 25$ by Lemma 2, and hence $M(14,8) = 24$. □

For $k = 7$, we have

$$3k - 1 + \lfloor \frac{k-1}{2} \rfloor = 21 - 1 + \lfloor \frac{7-1}{2} \rfloor = 23.$$

Hence, the conjecture $M(2k, k+1) = 3k - 1 + \lfloor \frac{k-1}{2} \rfloor$ does not hold for $k = 7$.

Example 2 Let

$$A_1 = \begin{bmatrix} J_{4,4} & J_{4,4} & 0 & J_{4,2} & 0 & 0 \\ J_{2,4} & 0 & J_{2,2} & J_{2,2} & 0 & J_{2,2} \\ J_{4,4} & 0 & J_{4,2} & 0 & J_{4,4} & 0 \\ 0 & J_{2,4} & J_{2,2} & J_{2,2} & 0 & J_{2,2} \\ 0 & J_{4,4} & 0 & 0 & J_{4,4} & J_{4,2} \\ 0 & 0 & J_{2,2} & J_{2,2} & J_{2,4} & J_{2,2} \end{bmatrix},$$

then $A_1 \in \Lambda(18, 10)$ and $b(A_1) = 32$, and it follows $M(18, 10) = 32$.

Proof By the same way as above, we can easily show $b(A_1) = 3$, and hence $s(A_1) = 36 - 1 - 3 = 32$. Thus, $M(18, 10) \geq 32$.

On the other hand, by Lemma 2 we have $M(18, 10) < 36 - 1 - \lceil \frac{18}{10} \rceil = 33$. Hence $M(18, 10) = 32$. □

Therefore, $M(2k, k+1) \neq 3k - 1 + \lfloor \frac{k-1}{2} \rfloor$ for $k = 9$ since $3k - 1 + \lfloor \frac{k-1}{2} \rfloor = 30$.

Note added in the proof: After the submission of this note, Professor R. A. Brualdi for his comments. Later, he introduced our examples in his column "From the Editor-in-Chief" (Linear Algebra and its Application, 320(2000), pp. 209-210) under our agreement. We have also found counter-examples to the conjecture for $k = 7, 9$. In fact, $M(12, 7) = 20$ and $M(16, 9) = 28$. As for $k \geq 11$, no counter-examples have been found to disprove this conjecture until now.

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猜想 $M(2k, k+1) = 3k - 1 + \lfloor \frac{k-1}{2} \rfloor$ 的反例

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摘 要: Brualdi 与 Jung 在 [1] 中研究了一类具有固定线和 k 的 $n \times n$ 矩阵上的最大跳跃数 $M(n, k)$, 并提出猜想

$$M(2k, k+1) = 3k - 1 + \lfloor \frac{k-1}{2} \rfloor.$$

本文给出了这一猜想的两个反例.