Journal of Mathematical Research & Exposition
Vol.22, No.2, 205-211, May, 2002

Quantum Groups by Ore Extensions Associated with Group Algebras *

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Abstract: As a continuation of the work of Beattie on quantum groups constructed by Ore extensions, in this paper, we characterize their centre and discuss the category of quantum Yang-Baxter modules over them. In addition, we determine all finite dimensional irreducible representations over these quantum groups.

Key words: Quantum group; centre; quantum Yang-Baxter module; irreducible representation.

Classification: AMS(2000) 16S40,16W30/CLC O153.3

Document code: A **Article ID:** 1000-341X(2002)02-0205-07

1. Introduction

Suppose that k is a field of characteristic zero and C is a cyclic group with generator c. Let $A = kC$ be the group algebra of C over k. According to Beattie [1,2], the Hopf algebra $(kC)_{\lambda,i}$ is defined as follows: where λ is a non-zero element in k and i is a fixed integer, as an algebra, the structure of $(kC)_{\lambda,i}$ coincides with the Ore extension $A[X,\phi]$ with derivation 0 (cf. [1]), where ϕ is the automorphism of algebra A determined by $\phi(c) = \lambda^{-1}c$ for any λ if C is infinite or λ is a primitive m-th root of unity when C has finite order m. The Hopf algebra structure on $A[X,\phi]$ is determined by

$$\triangle(c) = c\otimes c, \quad \triangle(X) = c^i\otimes X + X\otimes 1,$$

$$\varepsilon(c) = 1, \quad \varepsilon(X) = 0,$$

$$S(c) = c^{-1}, \quad S(X) = -c^{-i}X.$$

We note that when $\lambda \neq 1$ the Hopf algebra $(kC)_{\lambda,i}$ is non-commutative and non cocommutative. We call the Hopf algebra $(kC)_{\lambda,i}$ defined as above Beattie quantum group. Thus

*Received date: 1999-03-23
Foundation item: Supported by the National Natural Science Foundation of China (10071078) and the Young Teacher's Projects from the Chinese Education Ministry.
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we give infinitely many examples of quantum groups. In [1], Beattie has showed that the quantum group $(kC)_{\lambda,i}$ is pointed and not co-Frobenius [1, Proposition 4.1]. Moreover, if λ^i is a primitive n-th root unity, then the quantum group $(kC)_{n,\lambda,i} = (kC)_{\lambda,i}/(X^n)$ is pointed, co-Frobenius and not unimodular [1, Proposition 4.3], where (X^n) denotes the bi-ideal generated by X^n. The quantum groups $(kC)_{\lambda,i}$ and $(kC)_{n,\lambda,i}$ cover the examples in [1,2,13].

The main purpose of this paper is to give explicit description of the centre of $(kC)_{\lambda,i}$ and finite dimensional simple $(kC)_{\lambda,i}$-module. More precisely, in section 2, we show $(kC)_{\lambda,i}$ is a $\mathbb{Z}$-graded Hopf algebra, as a direct consequence, we obtain the result that $(kC)_{\lambda,i}$ is pointed. In addition, we give explicitly characterization of the centre of $(kC)_{\lambda,i}$. In section 3, by using the techniques similar to that in the reprsentation of Lie algebra [14], we give all finite dimensional simple $(kC)_{\lambda,i}$-modules. In section 4, we discuss the category of quantum Yang-Baxter $(kC)_{\lambda,i}$-modules (cf. [5,8,9,10]) when C is finite. For simplicity, throughout this paper, k denotes the field of complex numbers. We use the standard summatative notation in Sweedler's book [15] on Hopf algebra and the notation in Majid's book [6] on quantum group.

2. Some algebraic properties on $(kC)_{\lambda,i}$

We continue using the notation in [1] on $(kC)_{\lambda,i}$. Notice that the algebra $(kC)_{\lambda,i}$ can be regarded as an algebra with generators c and X satisfying the following relations

$$Xc = \lambda^{-1}cX.$$

In the case that C has order m, we take λ a primitive m-th root of unity. By a direct calculation, we have in $(kC)_{\lambda,i}$

$$X^s c^t = \lambda^{-ts} c^t X^s, \quad t \in \mathbb{Z}, \quad s \in \mathbb{N}. \tag{2.1}$$

We note that $(kC)_{\lambda,i}$ has a basis $\{c^t X^s\}$, where $s \in \mathbb{N}$ and $t \in \mathbb{Z}$ for C infinite or t is an integer between 0 and $m-1$ if C has order m. We also note that S^2 is the inner automorphism induced by c^{-i}, hence $S^2(X) = \lambda^{-i}X$. Therefore, when λ is not a root of unity, the order of S is infinite and when λ is a primitive m-th root of unity, then the order of S is $2m$.

Let q be a non-zero element in k. Recall [3,6] that the Gauss polymonial is defined by

$$\begin{pmatrix} n \\ k \end{pmatrix}_q = \frac{(n)!_q}{(k)!_q(n-k)!_q},$$

where $0 \le k \le n$. Using this polynomial, the comultiplication on basis elements in $(kC)_{\lambda,i}$ is given by

$$\triangle(c^t X^s) = \sum_{0\le k\le s} \begin{pmatrix} n \\ k \end{pmatrix}_{\lambda^{-i}} c^{t+ki} X^{s-k} \otimes c^t X^k. \tag{2.2}$$

In [1], Beattie has proved that H is pointed, i.e, Corad$(H) = G(H) = kC$, where $H = (kC)_{\lambda,i}$. Moreover, we have

Proposition 2.1 *The quantum group $H = (kC)_{\lambda,i}$ is a $\mathbb{Z}$-graded Hopf algebra.*

Proof According to the definition of the $\mathbb{Z}$-graded Hopf algebra [15], we shall prove that there exists a subspace sequence $\{A_n\}_{n\in\mathbb{N}}$ of H such that H is the direct sum of subspaces A_n, $H = \bigoplus_{n=0}^{\infty} A_n$, satisfying $A_k A_l \subset A_{k+l}$, $\triangle(A_n) \subset \sum_{k=1}^{n} A_k \otimes A_{n-k}$ and $\varepsilon(A_n) = 0$ for $n \geq 1$. Consider the subspace A_n generated by $\{yX^n\}_{n\in\mathbb{N}}$, where $y \in A = A_0$. It is readily checked that A_n satisfies the requested properties □

From the proof of Proposition 2.1, we have $A_0 = A = kC$. By [15, Proposition 11.1.1], we know $\mathrm{Corad}(H) \subset A_0$. Hence we have $kC = \mathrm{Corad}(H)$.

Now we consider the centre of $(kC)_{\lambda,i}$, we have

Proposition 2.2 *For any cyclic group C, let $H = (kC)_{\lambda,i}$ and $Z(H)$ be the centre of H. Then*

(1) If C is infinite, and λ is not a root of unity, then $Z(H) = k$;

(2) If C is infinite, λ a primitive m-th root of unity, then $Z(H) = k[c^m, c^{-m}, X^m]$, where $k[c^m, c^{-m}, X^m]$ denotes the subalgebra of H generated by c^m, c^{-m}, X^m;

(3) If C is finite with order m, then $Z(H) = k[X^m]$.

Proof First, let $u \in Z(H)$, then we write u as

$$u = \sum_{t,s} a_{ts} c^t X^s.$$

Since u commutes with c and X, we have

$$\sum_{t,s} a_{ts} c^{t+1} X^s = \sum_{t,s} a_{ts} \lambda^{-s} c^{t+1} X^s \tag{2.3}$$

and

$$\sum_{t,s} a_{ts} c^t X^{s+1} = \sum_{t,s} a_{ts} \lambda^{-t} c^t X^{s+1}. \tag{2.4}$$

Now we consider the following three cases.

(1) When C is infinte and λ is not a root unity. It is clear from the relations (2.3–2.4) that $u \in Z(H)$ if and only if $s = 0$ and $t = 0$, which implies $u \in k$.

(2) Suppose that C is infinite and λ is a primitive m-th root of unity. By relation (2.1), it is easy to check that c^m and X^m belong to the centre of H. Hence $k[c^m, c^{-m}, X^m] \subset Z(H)$. Again, let $u \in Z(H)$, from the relation (2.3–2.4), we immediately have $a_{ts} = 0$ except that $m|s$ and $m|t$, i.e., it follows that u is of the form

$$u = \sum_{t,s} a_{ts} c^{mt} X^m s \in k[c^m, c^{-m}, X^m].$$

Hence $Z(H) = k[c^m, c^{-m}, X^m]$.

(3) Suppose that C is finite with order m. An argument similar to that in (2) shows that $Z(H) = k[X^m]$. □

We note that λ^{-i} is also an m-th root of unity when λ is a primitive m-th root of unity. The Gauss polynomial $\binom{m}{k}_{\lambda^{-i}} = 0$ for $1 \leq k \leq m-1$. Hence from the relation (2.2) we have $\triangle(X^m) = X^m \otimes 1 + c^{mi} \otimes X^m \in Z(H) \otimes Z(H)$ and $\triangle(c^m) = c^m \otimes c^m \in Z(H) \otimes Z(H)$. Moreover, $S(c^m) = c^{-m}$, $S(X^m) = (-1)^m \lambda^{mi} c^{-mi} X^m \in Z(H)$, therefore we have

Corollary 2.3 *The centre $Z(H)$ of $H = (kC)_{\lambda,i}$ is a Hopf subalgebra of H.*

3. Finite dimentional simple $(kC)_{\lambda,i}$-module

In this section, we assume that $\lambda \neq 1$, and λ is a primitive m-th root of unity. If λ is not a root of unity, we set $m = \infty$. Our aim in this section is to determine all finite dimentional simple $(kC)_{\lambda,i}$-module by using the techniques similar to the weight theory of Lie algebra (cf. [14]).

For any $(kC)_{\lambda,i}$-module V and any scalar $\theta \neq 0$, denote by V_θ the eigen subspace of V for c, i.e., $V_\theta = \{v \in V | cv = \theta v\}$. We call θ a weight of V if $V_\theta \neq 0$, in this case $0 \neq v \in V_\theta$ is called weight vector with weight θ. A weight θ is said to be a highest weight with weight vector v if $Xv = 0$, the vector v is called a highest weight vector.

Lemma 3.1 *Suppose that V is a finite dimensional $(kC)_{\lambda,i}$-module of dimension $n < m$, then V contains a highest weight vector. Moreover, the action induced by X on V is nilpotent.*

Proof Since $k = \mathbf{C}$ is algebraically closed and V is a finite dimensional $(kC)_{\lambda,i}$-module, we know that there exists a non-zero vertor v and a scalar $\theta \neq 0$ such that $cv = \theta v$. If $Xv = 0$, then the vector v is a highest weight vector with weight θ. If $Xv \neq 0$, then for $0 \leq t \leq n$, we have

$$c(X^t v) = \lambda^t X^t (cv) = \theta \lambda^t (X^t v),$$

which implies that $X^t v$ is an eigenvector with eigenvalue $\theta\lambda^t$ for c. Since $n < m$, we get $n+1$ eigenvector with pairwise distinct eigenvalues for c. Hence, there exists an integer t such that $X^t v \neq 0$, $X^{t+1} v = 0$. Thus, the vector $X^t v$ is a highest weight vector with weight $\theta\lambda^t$.

We now prove that the action of X is nilpotent. Clearly, it suffices to check that 0 is the only possible eigenvalue of X. Suppose that v is a non-zero eigenvector for X with eigenvalue $b \neq 0$. By the relation $Xc = \lambda^{-1} cX$, for $0 \leq t \leq n$, $c^t v$ is a eigenvector with eigenvalue $\lambda^{-t} b$. Again, we have $n+1$ eigenvector with distinct eigenvalues for X, a contradiction. □

Theorem 3.2 *Suppose that V is a finite dimensional $(kC)_{\lambda,i}$-module with dimension $n < m$. If V is simple, then $n = 1$. In particular, when m is not a root of unity, any simple finite dimensional $(kC)_{\lambda,i}$-module is one dimensional.*

Proof According to Lemma 3.1, there exists a highest weight vector v with weight $\theta \neq 0$, that is, $cv = \theta v$ and $Xv = 0$. Clearly, $V' = kv$ is a submudule of V. Hence $V = kv$, which implies that V is one dimensional. □

Let us take any scalar $\theta \neq 0$ if C is infinite and θ be a primitive m-th root of unity if C is finite with order m. Consider the one dimensional vector $V = kv$ with basis v, define $cv = \theta v$ and $Xv = 0$. It is easy to check that V is a $(kC)_{\lambda,i}$-module. We denote the module defined above by V_θ. By Theorem 3.2, we see that any simple finite dimensional

$(kC)_{\lambda,i}$-module with dimension $< m$ is isomorphic V_θ. Moreover, up to isomorphism, when C is infinite, $(kC)_{\lambda,i}$ has infinitely many one dimensional modules. When C is finite with order m, then the number of one dimensional module is m. The following Lemma is well known

Lemma 3.3 *For any finite dimensional simple $(kC)_{\lambda,i}$-module, the action of the element in $Z(H)$ coincides with a saclar.*

Theorem 3.4 *Suppose that λ is a primitive m-th root of unity. Then there is no finite dimensional simple $(kC)_{\lambda,i}$-module with dimension $> m$.*

Proof We first consider the case in which there exists a non-zero eigenvector $v \in V$ for the action of c such that $Xv = 0$. Then the subspace V' generated by v is a submodule of V. A contradiction.

Now consider that there exists no non-zero eigenvector $v \in V$ for the action of c such that $Xv = 0$. Let v be a non-zero eigenvector with eigenvalue θ for the action of c. We have $Xv \neq 0$. We shall show the subspace V' generated v, Xv, $\cdots$, $X^{m-1}v$ is also a submodule with dimension $< m$. Clearly, V' is stable under c because $c(X^s v) = \lambda^s \theta X^s v \in V'$ for $0 \leq s \leq m-1$. Now, we show V' is stable under X. If $0 \leq s \leq m-2$, then $X(X^s v) = X^{s+1}v \in V'$. If $s = m-1$, we have $X(X^{m-1}v) = X^m v = \alpha v \in V'$ since X^m is in the centre of $(kC)_{\lambda,i}$ and by Lemma 3.3, where $\alpha \in k$. Indeed we also have $\alpha \neq 0$. If $\alpha = 0$, there would exist an integer $s \leq m-1$ such that $X^s v$ would be an eigenvector for c and $X^{s+1}v = 0$, which would contradict our assumption. □

Now we consider simple $(kC)_{\lambda,i}$-module with dimension m, when λ is a primitive m-th root of unity. We first construct a class of m-dimensional simple $(kC)_{\lambda,i}$-module. Let us consider an m-dimensional vector space V with a basis $\{v_0, v_1, \cdots, v_{m-1}\}$. We take any scalar $\theta \neq 0$ if C is infinite and θ is a primitive m-th root of unity if C is finite with order m. Define $cv_s = \theta\lambda^s v_s$ for $1 \leq s \leq m-1$, $Xv_{s-1} = v_s$ for $1 \leq s \leq m-2$, and $Xv_{m-1} = \gamma v_0$, where $\gamma \in k$. It is easy to check that V equipped the above action of $(kC)_{\lambda,i}$ is an m-dimensional simple $(kC)_{\lambda,i}$-module, denote this module by $V_{\theta,\gamma}$.

Theorem 3.5 *For any simple $(kC)_{\lambda,i}$-module V with dimension m, then there exist θ and γ in k such that V is isomorphic to $V_{\theta,\gamma}$.*

Proof Since there exists a nonzero eigenvector with eigenvalue $\theta \neq 0$ for the action of c, we write $cv_0 = \theta v_0$ for some $v_0 \neq 0$. In view of the fact that V is simple, we have $Xv_0 \neq 0$, otherwise $V' = kv_0$ is submodule of V. Using similar argument, we have $X^s v_0 \neq 0$ for $0 \leq s \leq m-1$. Thus $\{X^s v_0\}_{0\leq s\leq m-1}$ are non-zero eigenvectors of c with distinct eigenvalues $\{\lambda^s\theta\}_{0\leq s\leq m-1}$. Hence the vector sequence $\{X^s v_0\}_{0\leq s\leq m-1}$ is independent over k. Therefore V equals the vector space generated by v_0, Xv_0, $\cdots$, $X^{m-1}v_0$. We note also that $X^m v_0$ is an eigenvector of c with eigenvalue $\lambda^m\theta = \theta$, hence there exists a $\gamma \in k$ such that $X^m v_0 = \gamma v_0$. Thus the assertion is proved. □

We now suppose that λ^i is a primitive n-th root of unity. According to [1, Proposition 4.2], we know the ideal (X^n) generated by X^n is a bi-ideal. One can form a quantum group $(kC)_{n,\lambda,i} = (kC)_{\lambda,i}/(X^n)$. It is clear that a finite dimensional $(kC)_{n,\lambda,i}$-module is simple if and only if it is simple as $(kC)_{\lambda,i}$-module on which X^n acts by 0. By using

Theorem 3.2, Theorem 3.4 and Theorem 3.5, it is not difficult to prove the following

Proposition 3.6 *Any finite dimensional simple $(kC)_{n,\lambda,i}$-module is isomorphic to V_θ or $V_{\theta,0}$.*

4. Quantum Yang-Baxter $(kC)_{\lambda,i}$-module

Generally, let H be a Hopf algebra over a field k. a left quantum Yang-Baxter H-module (or left bicrossed H-module) introduced in [9,12] is a triple $(V,\cdot,\rho)$, where $(V,\cdot)$ is a left H-module and (V,ρ) is a right H-comodule, satisfying the following compatiblity condition

$$\sum h_1 v_0 \otimes h_2 v_1 = \sum (h_2 v)_0 \otimes (h_2 v)_1 h_1,$$

for all $h \in H$ and $v \in V$, where we write $\triangle(h) = \sum h_1 \otimes h_2$ and $\rho(v) = \sum v_0 \otimes v_1$. Denote by ${}_H\mathcal{Y}B^H$ the category of left quantum Yang-Baxter H-modules. It is well known that quantum Yang-Baxter modules are quite related to the solutions of the Yang-Baxter equation, low dimensional topology and knot theory. Recently, many papers are denoted to discuss these connections (c.f., e.g., [4,7,10,11]).

Suppose that H is a finite dimensional Hopf algebra over k. By [8, Proposition 4], we know the category ${}_H\mathcal{Y}B^H$ is isomorphic to the category of left modules over the Drinfel'd double $D(H)$.

Now suppose that Hopf algebra H has a bijection antipode S, denote by H° the dual Hopf algebra. Then $(H^\circ)^{cop}$ and H form matched a pair Hopf algebra in the sense of Majid[6]. We denote by $D_H(H^\circ)$ the double crossproduct $(H^\circ)^{cop} \bowtie H$ (c.f., [7]). By the argument similar to that in [5,7,8], we have following

Proposition 4.1 *Suppose that H is a Hopf algebra with bijection antipode S and H° is dense in H^*. Then the category ${}_H\mathcal{Y}B^H$ is isomorphic to the category of left rational $D_H(H^\circ)$-module, where left rational $D_H(H^\circ)$-module means that it is rational as left H°-module.*

We know a few examples of Hopf algebras in which H° is dense in H^* when H is infinite. Now we take $H = (kC)_{\lambda,i}$, where C is finite with order m. Then we have

Theorem 4.2 *Suppose that C is finite and $H = (kC)_{\lambda,i}$. Then H° is dense in H^*.*

Proof By Proposition 2.1, we know H is a $\mathbb{Z}$-graded Hopf algebra with H_n spanned by $\{yX^n\}_{y\in A}$, where $A = kC$. Since C is finite, H_n is finite dimensional. Hence H is a locally finite graded Hopf algebra. Therefore the graded dual Hopf algebra H^g is dense in H^* [15, section 11.2]. So the assertion follows from the fact that $H^g \subset H^\circ$. □

As a direct consequence of Theorm 4.2 and Proposition 4.1, we have

Corollary 4.3 *Suppose that C is finite, $H = (kC)_{\lambda,i}$. Then the category ${}_H\mathcal{Y}B^H$ can be identified with the category of left rational $D_H(H^\circ)$-module.*

We end this section by noting that $(kC)_{\lambda,i}$ is pointed for any cyclic group C, using the proof of [7, Proposition 11], it is easy to see that every simple object V in ${}_H\mathcal{Y}B^H$ has the form of Hv, where $\rho(v) = v \otimes y$ for some $y \in C$. A question for determining the dimension of simple objects in ${}_H\mathcal{Y}B^H$ needs further work.

Acknowledgements The authurs would like to take this opportunity to thank Prof. Zhang Pu for his constant comments.

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群代数 Ore 扩张的量子群

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摘 要: 本文讨论由 Beattie 通过群代数的 Ore 扩张所构造的量子群的中心和 Yang-Baxter 模范畴. 此外，完全刻画了这类量子群的不可约表示.