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Welding Problem of Two Different Orthotropic Elastic Strips *

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Abstract: In this paper,welding problem of two orthotropic elastic strips with dissimilar materials is studied. By means of plane elastic complex method and theory of integral equation, a new algorithm is given, which improves the usual method of purely integral transformation.Theoretically, the stress distribution is obtained in a closed form.

Key words: orthotropic strip; welding; stress functions; plane elastic complex variable method.

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Plane elastic complex method and theory of integral equation are powerful tools in solution the problem of elastic mechanics. There are some classical literatures which due to Muskhelishvili $N.I.$[1], Lekhnitskii S.G.[2] and Lu Jianke[3]. In recent years, some significance works ([5,6,7]) have been done on the welding problems of orthotropic materials.

This paper, the welding problem of two orthotropic elastic strips with dissimilar materials is studied. First, the stress functions are decomposed in due form by way of an constructivity integral transformation.Then,the problem is reduced to a special system of integral equations by analytic method. Finally, the system of integral equations is solved and the solution of stress functions is obtained in a closed form.

1. Fundmental knowledge

For the elastic plane problem of orthosotropic materials,the stress and displacement can be expressed by two functions $\Phi(z_1)$ and $\Psi(z_2)$ [2]

$$\begin{cases} \sigma_x = 2\mathrm{Re}[\lambda_1^2\Phi'(z_1) + \lambda_2^2\Psi'(z_2)], \\ \sigma_y = 2\mathrm{Re}[\Phi'(z_1) + \Psi'(z_2)], \\ \tau_{xy} = -2\mathrm{Re}[\lambda_1\Phi'(z_1) + \lambda_2\Psi'(z_2)] \end{cases} \tag{1.1}$$

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and

$$\begin{cases} u_x = 2\mathrm{Re}[p_1\Phi(z_1) + p_2\Psi(z_2)], \\ u_y = 2\mathrm{Re}[q_1\Phi(z_1) + q_2\Psi(z_2)], \end{cases} \tag{1.2}$$

where $z_j = x + \lambda_j y, p_j = a_{11}\lambda_j^2 + 2a_{12} - a_{16}, q_j = a_{12}\lambda_j + \frac{a_{22}}{\lambda_j} - a_{26}, (j = 1, 2)$ are elastic constants.

In the case where the external forces X_n, Y_n are given on the boundary L, we have the boundary condition on L:

$$\begin{cases} 2\mathrm{Re}[\Phi(z_1) + \Psi(z_2)] = -\int Y_n ds + c_1, \\ 2\mathrm{Re}[\lambda_1\Phi(z_1) + \lambda_2\Psi(z_2)] = \int X_n ds + c_2, \end{cases} \quad z \in L. \tag{1.3}$$

In the case where the displacement $u(z) + iv(z)$ are given on the boundary L, we have the boundary condition on L:

$$\begin{cases} 2\mathrm{Re}[p_1\Phi(z_1) + p_2\Psi(z_2)] = u(z), \\ 2\mathrm{Re}[q_1\Phi(z_1) + q_2\Psi(z_2)] = u(z). \end{cases} \quad z \in L. \tag{1.4}$$

2. Welding problem of two strips

Let the elastic body consist of two different orthotropic elastic strips ,which occupy the ribbon-region $S_1(|x| < +\infty, 0 < y < a)$ and $S_2(|x| < +\infty, -b < y < 0)$ respectively,and corresponding elastic coefficients are $a_{jk}^1, a_{jk}^2 (j, k = 1, 2, 6)$ and $p_{1j}, p_{2j}, q_{1j}, q_{2j}$ (j=1,2,as the same meaning as in following sections).With no initial stress, the two strips is welded along the X-axis(i.e. real-axis) and bonded completely(similar to discuss as it exists different displacement).Donote $L_1(|x| < +\infty, y = a)$ and $L_2(|x| < +\infty, y = -b)$ as the other boundary of the strip ,there are external stress $X_j(t) + iY_j(t)$ acting on L_i,which principl vectors is $X_j + iY_j = \int_{L_i}[X_j(t) + iY_j(t)]dt$ and satisfied $\sum_{j=1}^{2}(X_j + iY_j) = 0$. Knowing stress and rotation on the infinite ,Let's study the statics balance of elastic body.

Without loss of generality,we can assume $X_j + iY_j = 0$,and no stress and rotation on the infinite. Thus, the stress funstions are single-valued holomorphic on the elastic body. Affine transformations $z_j = x + \lambda_{1j}y$ and $z_j = x + \lambda_{2j}y$ change the strip S_j to the strip S_{1j} and S_{2j} respectively, and the boundary L_j change to the boundary L_{1j} and L_{2j}, but the X-axis is unchanged.

According to the known conditions for the previous, our problem is attributed to solve the following boundary value problem

$$A_j\Phi_j(t_1) + B_j\overline{\Phi_j(t_1)} + \Psi_j(t_2) = F_j(t), t \in L_j, t_1 \in L_{j1}, t_2 \in L_{j2}, \tag{2.1}$$

$$\begin{aligned} &\alpha_{11}\Phi_1(x) + \alpha_{12}\overline{\Phi_1(x)} + \alpha_{13}\Psi_1(x) + \alpha_{14}\overline{\Psi_1(x)} \\ &\quad = \alpha_{21}\Phi_2(x) + \alpha_{22}\overline{\Phi_2(x)} + \alpha_{23}\Psi_2(x) + \alpha_{24}\overline{\Psi_2(x)}, \quad x \in X, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\beta_{11}\Phi_1(x) + \beta_{12}\overline{\Phi_1(x)} + \beta_{13}\Psi_1(x) + \beta_{14}\overline{\Psi_1(x)} \\ &\quad = \beta_{21}\Phi_2(x) + \beta_{22}\overline{\Phi_2(x)} + \beta_{23}\Psi_2(x) + \beta_{24}\overline{\Psi_2(x)}, \quad x \in X, \end{aligned} \tag{2.3}$$

where $\Phi_j(z_1)$ and $\Psi_j(z_2)$ are stress functions on S_j.

$$(A_1, B_1, A_2, B_2) = (\frac{\lambda_{11} - \overline{\lambda_{12}}}{\lambda_{12} - \overline{\lambda_{12}}}, \frac{\overline{\lambda_{11} - \lambda_{12}}}{\lambda_{12} - \overline{\lambda_{12}}}, \frac{\lambda_{21} - \overline{\lambda_{22}}}{\lambda_{22} - \overline{\lambda_{12}}}, \frac{\overline{\lambda_{21} - \lambda_{22}}}{\lambda_{22} - \overline{\lambda_{22}}}),$$

$$\begin{aligned}&(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24})\\ &\quad = (1 + i\lambda_{11}, 1 + i\overline{\lambda_{11}}, 1 + i\lambda_{12}, 1 + i\overline{\lambda_{12}}, 1 + i\lambda_{21},\\ &\qquad 1 + i\overline{\lambda_{21}}, 1 + i\lambda_{22}, 1 + i\overline{\lambda_{22}}),\end{aligned}$$

$$\begin{aligned}&(\beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{21}, \beta_{22}, \beta_{23}, \beta_{24})\\ &\quad = (p_{11} + iq_{11}, p_{11} + i\overline{q_{11}}, p_{12} + iq_{12}, p_{12} + i\overline{q_{12}}, p_{21} + iq_{21},\\ &\quad p_{21} + i\overline{q_{21}}, p_{22} + iq_{22}, p_{22} + i\overline{q_{22}}),\end{aligned}$$

$$\begin{aligned}F_j(t) &= \frac{(1 - i\overline{\lambda_{j2}})f_j(t) - (1 + i\overline{\lambda_{j2}})\overline{f_j(t)}}{2i(\lambda_{j2} - \overline{\lambda_{j2}})} + C_j, f_j(t)\\ &= i\int_{-\infty + ri}^{t} [X_j(t) + iY_j(t)]dt, t \in L_j \quad (j = 1, r = a; j = 2, r = -b).\end{aligned}$$

3. The method of solving the stress functions

Defination 1

$$w_j(\tau) = \Phi_j(\tau_1) \in H,\ \tau \in L_j,\ \tau_1 \in L_{j1}. \tag{3.1}$$

From Eq.(2.3),we have

$$\Psi_j(\tau_2) = F_j(\tau) - A_j w_j(\tau) - B_j \overline{w_j(\tau)},\ \tau \in L_j, \tau_2 \in L_{j2}. \tag{3.2}$$

Defination 2

$$\Phi_{j0}(z) = \frac{1}{2\pi i}\int_{L_{j1}} \frac{w_j(\tau)}{\tau_1 - z} d\tau_1,\quad z \notin L_{j1}, \tag{3.3}$$

$$\Psi_{j0}(z) = \frac{1}{2\pi i}\int_{L_{j2}} \frac{F_j(\tau) - A_j w_j(\tau) - B_j \overline{w_j(\tau)}}{\tau_1 - z} d\tau_2,\quad z \notin L_{j2}. \tag{3.4}$$

Obviously, $\Phi_{j0}(z)$ and $\Psi_{j0}(z)$ are sectionally holomorphic functions with line of jump L_{1j} and L_{2j} respectively and $\Phi_{j0}(\infty) = \Psi_{j0}(\infty) = 0$. Following Eqs.(3.1)- (3.4) and Plemelj's formula, we have

Lemma 3

$$\begin{cases} \Phi^+_{j0}(\tau) = \Phi_j(\tau) + \Phi^-_{j0}(\tau), & \tau \in L_{j1},\\ \Psi^+_{j0}(\tau) = \Psi_j(\tau) + \Psi^-_{j0}(\tau), & \tau \in L_{j2}.\end{cases} \tag{3.5}$$

Defination 4

$$\Phi_{j1}(z) = \begin{cases} \Phi_{j0}(z) + \Phi_j(z), & z \in S_{j1},\\ \Phi_{j0}(z), & z \in Z_j - S_{j1}(Z_1 = Z^+, Z_2 = Z^-),\end{cases} \tag{3.6}$$

$$\Psi_{j1}(z) = \begin{cases} \Psi_{j0}(z) + \Psi_j(z), & z \in S_{j1}, \\ \Psi_{j0}(z), & z \in Z_j - S_{j2}(Z_1 = Z^+, Z_2 = Z^-). \end{cases} \tag{3.7}$$

Following Defination 4 and Lemma 3, easy to know that $\Phi_{11}(z), \Psi_{11}(z)$ and $\Phi_{21}(z), \Psi_{21}(z)$ are holomorphic functions on upper semi-plane and lower semi-plane respectively, both continuously extend to boundary X-axis.

Substituting Eqs(3.6)-(3.7) to Eqs.(2.1)-(2.3) respectively, we obtain

$$\begin{aligned} &A_j[\Phi_{j1}(t_1) - \Phi_{j0}(t_1)] + B_j[\overline{\Phi_{j1}(t_1) - \Phi_{j0}(t_1)}] + \Psi_{j1}(t_2) - \Psi_{j0}(t_2) \\ &\quad = F_j(t), \quad t \in L_j, t_1 \in L_{j1}, t_2 \in L_{j2}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\alpha_{11}[\Phi_{11}(x) - \Phi_{10}(x)] + \alpha_{12}[\overline{\Phi_{11}(x) - \Phi_{10}(x)}] + \\ &\quad \alpha_{13}[\Psi_{11}(x) - \Psi_{10}(x)] + \alpha_{14}[\overline{\Psi_{11}(x) - \Psi_{10}(x)}] \\ &\quad = \alpha_{21}[\Phi_{21}(x) - \Phi_{20}(x)] + \alpha_{22}[\overline{\Phi_{21}(x) - \Phi_{20}(x)}] + \\ &\quad \alpha_{23}[\Psi_{21}(x) - \Psi_{20}(x)] + \alpha_{24}[\overline{\Psi_{21}(x) - \Psi_{20}(x)}], \quad x \in X, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\beta_{11}[\Phi_{11}(x) - \Phi_{10}(x)] + \beta_{12}[\overline{\Phi_{11}(x) - \Phi_{10}(x)}] + \\ &\quad \beta_{13}[\Psi_{11}(x) - \Psi_{10}(x)] + \beta_{14}[\overline{\Psi_{11}(x) - \Psi_{10}(x)}] \\ &\quad = \beta_{21}[\Phi_{21}(x) - \Phi_{20}(x)] + \beta_{22}[\overline{\Phi_{21}(x) - \Phi_{20}(x)}] + \\ &\quad \beta_{23}[\Psi_{21}(x) - \Psi_{20}(x)] + \beta_{24}[\overline{\Psi_{21}(x) - \Psi_{20}(x)}], \quad x \in X. \end{aligned} \tag{3.10}$$

Following Eqs (3.9)-(3.10) and their conjugate equations,by help of formula of Cauchy-pattern integral on X-axis,we have

Lemma 5

$$\begin{cases} \Phi_{11}(z) = s_{11}\overline{\Phi_{10}(\overline{z})} + s_{12}\overline{\Psi_{10}(\overline{z})} + s_{13}\Phi_{20}(z) + s_{14}\Psi_{20}(z) \\ \Psi_{11}(z) = s_{21}\overline{\Phi_{10}(\overline{z})} + s_{22}\overline{\Psi_{10}(\overline{z})} + s_{23}\Phi_{20}(z) + s_{24}\Psi_{20}(z) \\ \Phi_{21}(z) = s_{31}\overline{\Phi_{10}(\overline{z})} + s_{32}\overline{\Psi_{10}(\overline{z})} + s_{33}\Phi_{20}(z) + s_{34}\Psi_{20}(z) \\ \Psi_{21}(z) = s_{41}\overline{\Phi_{10}(\overline{z})} + s_{42}\overline{\Psi_{10}(\overline{z})} + s_{43}\Phi_{20}(z) + s_{44}\Psi_{20}(z) \end{cases} \quad \mathrm{Im}z > 0,$$

where $s_{ij} = s_{ij}(\alpha_{ki}, \overline{\alpha_{ki}}, \beta_{ki}, \overline{\beta_{ki}})(k = 1,2; i,j,l = 1,2,3,4)$ can be determined easily.

For Im$z < 0$,we have the same conclution,but now the coefficients s_{ij} is changed to r_{ij},which to be determined by $\alpha_{ki}, \overline{\alpha_{ki}}, \beta_{ki}, \overline{\beta_{ki}}$ similiar.

From lemma 5, Defination 2 and Eq (3.8),by plemelj's integral formula, Cauchy-pattern integral inversion formula and Poincare'-Bertrand's transform *formula*[4],we obtain following two equations associated with $j = 1,2$ respectively

$$\begin{aligned} &w_{1^*}(x_0) + \int_{-\infty}^{+\infty} k_{11}(x, x_0)w_{1^*}(x)\mathrm{d}x + \int_{-\infty}^{+\infty} k_{12}(x, x_0)\overline{w_{1^*}(x)}\mathrm{d}x + \\ &\int_{-\infty}^{+\infty} k_{13}(x, x_0)w_{2^*}(x)\mathrm{d}x + \int_{-\infty}^{+\infty} k_{14}(x, x_0)\overline{w_{2^*}(x)}\mathrm{d}x = g_1(x_0), \end{aligned} \tag{3.11}$$

$$w_{2^*}(x_0) + \int_{-\infty}^{+\infty} k_{21}(x,x_0)w_{1^*}(x)\mathrm{d}x + \int_{-\infty}^{+\infty} k_{22}(x,x_0)\overline{w_{1^*}(x)}\mathrm{d}x +$$

$$\int_{-\infty}^{+\infty} k_{23}(x,x_0)w_{2^*}(x)\mathrm{d}x + \int_{-\infty}^{+\infty} k_{24}(x,x_0)\overline{w_{2^*}(x)}\mathrm{d}x = g_2(x_0), \qquad (3.12)$$

where $w_{1^*}(x) = w(x+ai), w_{2^*}(x) = w(x-bi)$ and $g_1(x), g_2(x), k_{jl}(x,x_0)$(j=1,2,l=1,2,3,4) can be determined easily.

By the uniqueness theorem[3],as the same method as in [3],we can prove the Eqs.(3.11)-(3.12) are uniquely solvable when undetermined constants C_1 and C_2 are suitably chosen.

From Eqs.(3.11)-(3.12) and their conjugate equations, we obtain following system of integral equations

$$M(x_0) + \int_{-\infty}^{+\infty} M(x)K(x,x_0)\mathrm{d}x = G(x_0), \qquad (3.13)$$

where

$$M(x) = (w_{1^*}(x), \overline{w_{1^*}(x)}, w_{2^*}(x), \overline{w_{2^*}(x)}), G(x) = (g_1(x), \overline{g_1(x)}, g_2(x), \overline{g_2(x)}),$$

$$K(x,x_0) = \begin{pmatrix} k_{11}(x,x_0) & \overline{k_{12}(x,x_0)} & k_{21}(x,x_0) & \overline{k_{22}(x,x_0)} \\ k_{12}(x,x_0) & \overline{k_{11}(x,x_0)} & k_{22}(x,x_0) & \overline{k_{21}(x,x_0)} \\ k_{13}(x,x_0) & \overline{k_{14}(x,x_0)} & k_{23}(x,x_0) & \overline{k_{24}(x,x_0)} \\ k_{14}(x,x_0) & \overline{k_{13}(x,x_0)} & k_{24}(x,x_0) & \overline{k_{23}(x,x_0)} \end{pmatrix}.$$

With the theory of integral *equation*[8], we get the solution of the system integral equations(3.13) as follows

$$M(x_0) = G(x_0) + \int_{-\infty}^{+\infty} G(x)\Gamma(x,x_0;1)\mathrm{d}x. \qquad (3.14)$$

Hence, $\Gamma(x,x_0;1)$ is the solution kernel matrix of kernel matrix $K(x,x_0)$.

As constant don't effect on the stress of materials,the constants C_j metion previous can be determined at will.From define2,define4 and lemma5, the solution of the stress functions $\Phi_j(z_1)$ and $\Psi_j(z_2)$ is given in closed from.

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两个不同正交各向异性弹性长条的焊接问题

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摘 要: 本文利用弹性复变方法和积分方程理论，讨论两个不同材料正交各向异性弹性长条的焊接问题，给出一种新的算法，改进通常单一的积分变换方法，理论上，得出了应力分布封闭形式的解.