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Boundedness of the Equiconvergent Operators on the Sphere *

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Abstract: Let $\mathbf{R}^n$ be an n-dimensional Euclidean space with $n \geq 3$. Denote by Ω_n the unit sphere in $\mathbf{R}^n$. For a function $f \in L(\Omega_n)$ we denote by $E_N^\delta(f)$ the equiconvergent operator of Cesàro means of order δ of the Fourier-Laplace series of f. The special value $\lambda := \frac{n-2}{2}$ of δ is known as the critical index. For $0 < \delta \leq \lambda$, we set $p_0 := \frac{2\lambda}{\lambda+\delta}$. The main aim of this paper is to prove that

$$\|E_N^\delta(f)\|_{p_0} \leq C_{(n,\ l)}\Big(\int_{\Omega_n} |f(x)|^{p_0}(\log^+|f(x)|)^{2-p_0}(\log^+\log^+|f(x)|)^{l(p_0-1)}\,\mathrm{d}x + 1\Big),$$

with $l > 1$.

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1. Introduction and main results

Let f be an integrable function on the unit sphere $\Omega_n := \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 = 1\}$ of $\mathbf{R}^n$ and let $\sigma(f)(x) := \sum_{k=0}^{\infty} Y_k(f)(x)$ be the Fourier-Laplace expansion of f, where Y_k is the projection operator from $L(\Omega_n)$ to the space of all spherical harmonics of degree k. The equiconvergent operator E_N^δ of order δ ($\mathrm{Re}\delta > -1$) is the linear means defined by

$$E_N^\delta(f)(x) := \sum_{k=0}^{N} b_{n,k}^\delta Y_k(f)(x), \tag{1}$$

where

$$b_{N,k}^\delta := \frac{\Gamma(N+n-1)\Gamma(N+\delta+k+n-1)A_{N-k}^\delta}{\Gamma(N+k+n-1)\Gamma(N+\delta+n-1)A_N^\delta}.$$

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The equiconvergent operator E_N^δ on the sphere was first introduced by Wang[4] and has been studied by several authors (see for example [2] and [4]). The main purpose of this paper is to prove

Theorem *Let* $0 < \delta \le \lambda := \frac{n-2}{2}$ *and* $p_0 := \frac{2\lambda}{\lambda+\delta}$, *then*

$$\|E_N^\delta(f)\|_{p_0} \le C_{(n,\,l)}\Big(\int_{\Omega_n} |f(x)|^{p_0}(\log^+|f(x)|)^{2-p_0}(\log^+\log^+|f(x)|)^{l(p_0-1)}\,dx + 1\Big),$$

with $l > 1$.

Our Theorem here can be considered as an extension of the result in [2] which says that $\sup\limits_N \|E_N^\delta\|_{(p,\,p)} < \infty$ whenever $p > p_0$.

The paper is organized as follows: In §2 we discribe some basic facts about the equiconvergent operator and Jacobi polynomials. In §3 we prove our Theorem by using Stein's interpolation theorem and the method of expolation.

2. Preliminary propositions

Proposition 1 *The Equiconvergent operator* E_N^δ *can be represented as*

$$E_N^\delta(f)(x) = \gamma_N^\delta \int_{\Omega_n} f(y) P_N^{(\frac{n-1}{2}+\delta,\frac{n-3}{2})}(x\cdot y)\,dy,\quad \mathrm{Re}\delta > -1, \tag{2}$$

where $\gamma_N^\delta := \frac{\Gamma(\delta+1)\Gamma(N+1)\Gamma(N+n-1)}{(4\pi)^{\frac{n-1}{2}}\Gamma(N+\delta+1)\Gamma(N+\frac{n-1}{2})}$ *and* $P_k^{(\alpha,\beta)}$ *denotes Jacobi polynomial with complex indexes defined in [2].*

Obviously $\{E_N^\delta :\quad -1 < \mathrm{Re}\,\delta \le n\}$ is a family of analytic operators.

Proposition 2 *Let* $\alpha \in [0, 2n]$, $\beta \in [0, n]$ *and* $\tau \in \mathbf{R}$, *then*

$$|P_k^{(\alpha+\frac{1}{2}+i\tau,\beta)}(\cos\theta)| \le \begin{cases} c_n e^{3|\tau|}k^{\alpha+1}, & 0<\theta<2k^{-1},\\ c_n e^{3|\tau|}k^{-\frac{1}{2}}\theta^{-\alpha-1}(\pi-\theta)^{-\beta-1}, & 2k^{-1}<\theta<\pi-2k^{-1},\\ c_n e^{3|\tau|}k^{\beta+\frac{1}{2}}, & \pi-2k^{-1}<\theta<\pi. \end{cases} \tag{3}$$

Propositions 1 and 2 are contained in [2] and [4], respectively.

3. The Lemmas

To establish our Theorem, we need a series of lemmas.

Lemma 1 *Let* $\varepsilon \in (0, n)$ *and* $\alpha = \lambda + \varepsilon + i\tau$ *with* $\tau \in \mathbf{R}$, *then*

$$\|E_N^\alpha(f)\|_1 \le c_n \varepsilon^{-1} e^{5\tau}\|f\|_1.$$

Proof It follows from (2) that

$$E_N^\alpha(f)(x) = |\Omega_{n-2}|\gamma_N^\alpha \int_0^\pi S_\theta(f)(x) P_N^{(n-\frac{3}{2}+\varepsilon+i\tau,\frac{n-3}{2})}(\cos\theta)\sin^{n-2}\theta\, d\theta,$$

where S_θ is the spherical translation operator which is defined by

$$S_\theta(f)(x) := \frac{1}{|\Omega_{n-2}|}\int_{\{y\in\Omega_n: x\cdot y=0\}} f(x\cos\theta + y\sin\theta)\,\mathrm{d}l(y), \quad 0<\theta<\pi, \quad f\in L(\Omega_n),$$

where $\mathrm{d}l(y)$ denotes the measure elements on $\{y\in\Omega_n : x\cdot y = 0\}$.

Noticing that $|\gamma_N^\alpha| \le c_n e^{\pi|\tau|}N^{\frac{1}{2}-\epsilon}$, on account of Minkovski inequality and the fact that $\|S_\theta\|_{(1,1)} = 1$, we have

$$\begin{aligned} E_N^\alpha(f)\|_1 &\le c_n e^{\pi|\tau|}N^{\frac{1}{2}-\epsilon}\|f\|_1\cdot[\int_0^{N^{-1}} + \int_{N^{-1}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}]|P_N^{(n-\frac{3}{2}+\epsilon+i\tau,\frac{n-3}{2})}(\cos\theta)|\sin^{n-2}\theta\,\mathrm{d}\theta \\ &\overset{\mathrm{def}}{=} c_n e^{\pi|\tau|}\|f\|_1\cdot[I_N^{(1)} + I_N^{(2)} + I_N^{(3)}] \end{aligned}$$

However, by (3) and elementary calculations, we get

$$I_N^{(i)} \le B_n e^{5|\tau|}\epsilon^{-1}, \quad i=1,2,3.$$

Putting these together, we get Lemma 1.

Lemma 2 *Let* $0\le\sigma\le n$ *and* $\alpha=\sigma+i\tau$ *with* $\tau\in\mathbf{R}$, *then*

$$\|E_N^\alpha(f)\|_2 \le Ce^{\pi|\tau|}\|f\|_2$$

Proof It follows from (1) that

$$\|E_N^\alpha(f)\|_2^2 = \sum_{k=0}^N |b_{N,k}^\alpha|^2\|Y_k(f)\|_2^2$$

Therefore, we only need to prove that

$$\sup_{N,k}|b_{N,k}^\alpha| \le c_n e^{\pi|\tau|}.$$

However, this follows immediatly from the following two inequalities:

$$\sqrt{1+\frac{v^2}{u^2}} \le \frac{\Gamma(u)}{|\Gamma(u+iv)|} \le \sqrt{1+\frac{v^2}{u^2}}e^{\frac{\pi}{2}|v|}, \quad (u>0,\ v\in\mathbf{R}),$$
$$e^{-1-\gamma v}(u+v)^v \le \frac{\Gamma(u+v)}{\Gamma(u)} \le e^{(1-\gamma)v}(u+v)^v, \quad (u\ge 1, v>0),$$

where γ is the Euler constant. □

Lemma 3 *Let* $0<\delta<\lambda$ *and* $p_0 := \frac{2\lambda}{\lambda+\delta}$, *then*

$$\|E_N^\delta\|_{p,p} \le c_n(\frac{1}{p-p_0})^{\frac{\delta}{\lambda}}, \quad (p_0<p\le 2).$$

Proof Let $\varepsilon = \frac{(p-p_0)(\lambda+\delta)}{2-p}$ and $t = \frac{\delta}{\lambda+\varepsilon}$, then by elementary calculations we know that

$$\frac{1}{p} = \frac{1-t}{2} + t \quad \text{and} \quad \delta = (\lambda+\varepsilon)t + 0(1-t).$$

Thus, applying Stein's interpolation theorem for analytic families of operators and taking into account Lemmas 1 and 2, we derive Lemma 3.

4. Proof of the Theorem

Let us write

$$f_k(x) := \begin{cases} f(x), & x \in E_k, \\ 0 & \text{otherwise}, \end{cases}$$

where $E_0 = \{x : |f(x)| < 2\}$ and $E_k = \{x : 2^{2^{k-1}} \le |f(x)| < 2^{2^k}\}$ with $k = 1, 2\cdots$. Then on account of Minkovski inequality, we have

$$\|E_N^\delta(f)\|_{p_0} \le \sum_{k=0}^{\infty} \|E_N^\delta(f_k)\|_{p_0}.$$

However, by Hölder inequality and Lemma 3, we have

$$\|E_N^\delta(f_k)\|_{p_0} \le c_n \|E_N^\delta(f_k)\|_2 \le c_n \|f_0\|_2 \le c_n$$

and

$$\begin{aligned} \|E_N^\delta(f_k)\|_{p_0} &\le c_n \|E_N^\delta(f_k)\|_{p_k} \le c_n 2^{k\frac{\delta}{\lambda}} \|f_k\|_{p_k} \\ &\le c_n \Big(\int_{E_k} |f|^{p_0} (\log|f|)^{2-p_0}\, dx\Big)^{\frac{1}{p_k}}, \end{aligned}$$

where $p_k = p_0 + 2^{-k}$ with $k \ge 1$.

On the other hand , by Young inequality, it follows that for any $l > 1$,

$$\begin{aligned} &\Big(\int_{E_k} |f(x)|^{p_0} (\log|f(x)|)^{2-p_0}\, dx\Big)^{\frac{1}{p_k}} = k^{-l\frac{p_0-1}{p_0}} k^{l\frac{p_0-1}{p_0}} \Big(\int_{E_k} |f(x)|^{p_0} (\log|f(x)|)^{2-p_0}\, dx\Big)^{\frac{1}{p_k}} \\ &\le c_{n,\delta}\Big(k^{-l} + \int_{E_k} |f(x)|^{p_0} (\log|f(x)|^{2-p_0} (\log\log|f(x)|)^{l(p_0-1)}\, dx\Big). \end{aligned}$$

Putting these together, we establish our theorem.

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球面上等收敛算子的有界性

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摘 要: 设 $\mathbf{R}^n$ 是 n- 维欧氏空间 $n \geq 3$. 用 Ω_n 表示 $\mathbf{R}^n$ 上的单位球面，对于函数 $f \in L(\Omega_n)$, $E_N^\delta(f)$ 表示其 Fourier-Laplace 级数的 δ 阶 Cesàro 平均所决定的等收敛算子，其中，$\lambda := \frac{n-2}{2}$ ，δ 是熟知的临界指标. 对于 $0 < \delta \leq \lambda$, 令 $p_0 := \frac{2\lambda}{\lambda+\delta}$, 本文主要证明了如下结果：

$$\|E_N^\delta(f)\|_{p_0} \leq C_{(n,\,l)}\Big(\int_{\Omega_n} |f(x)|^{p_0}(\log^+|f(x)|)^{2-p_0}(\log^+\log^+|f(x)|)^{l(p_0-1)}\,\mathrm{d}x + 1\Big),\ l > 1.$$