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Limit Theorems of a Class of Super-Uniformly Elliptic Diffusions *

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Abstract: In this paper, we investigate super-uniformly elliptic diffusions $\{X_t, t \geq 0\}$ with its branching mechanism given by $\psi(z) = \gamma z^{1+\beta}(0 < \beta \leq 1)$, and, when the initial value $X_0(dx)$ is one kind of invariant measures of the underlying processes, we show that if dimension d satisfies $\beta d \leq 2$, then the random measures X_t will converge to the null in distribution and if $\beta d > 2$, then X_t will converge to a nondegenerative random measure in the same sense.

Key words: uniformly elliptic diffusion; superproces; limit theorem.

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1. Introduction and main results

Let ξ_t be the uniformly elliptic diffusion processes generated by the infinitesimal differential operator L, which is defined as follows:

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i},$$

where the coefficients $a_{ij}(x)(i,j = 1,\cdots,d)$ and $b_i(x)(i = 1,\cdots,d)$ satisfy uniformly elliptic conditions. Additionally, we assume that the processes satisfy the following:

Condition (H)

(H1) The transition function $p(t,x,y)$ of ξ_t satisfies that : $p(t,x,y) > 0$ and $M_1 t^{-\frac{d}{2}} \exp(-c_1(|y-x|^2/t)) - M_2 t^{-\frac{d}{2}+r}\exp(-c_2(|y-x|^2/t) \leq p(t,x,y) \leq Mt^{-\frac{d}{2}}\exp(-\alpha(|y-x|^2/t))$ for $\forall t > 0, x, y \in R^d$, where $M, M_1, M_2, \alpha, c_1, c_2$ and r are all positive constants.

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(H2) There exist some invariant measures of ξ_t which have the following form: $\mu(dx) = m(x)\mathrm{d}x$, where $dx = \lambda(dx)$ is the Lebesgue measure and $0 < m(x) < P$ for some constant P. Without loss of generality, we suppose that $P \geq 1$.

The existence of the process ξ_t can be found in [1], [2] and [3].

Let $M_p(R^d)$ be the set of all Radon measures on R^d and such that $\int_{R^d} \frac{1}{1+\|x\|^p}\nu(\mathrm{d}x) < \infty$ for $p \geq d$. Denote by $pC(R^d)$ (resp. $pC_c(R^d)$) the space of positive continuous functions (resp. with compact support) on R^d. The so-called super-uniformly elliptic diffusions are the Markov processes $\{X_t, P^\mu\}_{\mu\in M_p(R^d)}$ with its Laplace functionals given by

$$E^\mu e^{-\langle f, X_t\rangle} = e^{-\langle u_t, \mu\rangle}, \quad f \in pC_c(R^d), \mu \in M_p(R^d),$$

where u_t is the unique mild solution of the following differential equation

$$\frac{\partial u(t,x)}{\partial t} = Lu(t,x) - \psi(x, u(t,x)), \quad u(0,x) = f(x), \tag{1}$$

where L is the infinitesimal operator of ξ_t, and $\psi(x,z) = \gamma z^{1+\beta}$ is called branching mechanism. The equation (1) is equivalent to the following integral equation

$$u_t(x) = P_t f - \int_0^t P_{t-s}\psi(u_s)\mathrm{d}s. \tag{2}$$

For the existence of X_t, we refer the reader to [4-7].

Our main results are:

Theorem 1.1 *Let ξ_t satisfy the condition (H), X_t be the corresponding superprocesses with branching mechanism given by $\psi(z) = \gamma z^{1+\beta}$, and $X_0(\mathrm{d}x) = \mu(\mathrm{d}x)$ be an invariant measure of ξ_t. If dimension d of the underlying space such that $\beta d > 2$, then X_t will convergence to a nondegenerative random measure X_∞, namely, $X_t \xrightarrow{d} X_\infty$ as $t \to \infty$.*

Theorem 1.2 *Assume ξ_t and X_t be as in Theorem 1.1. If $\beta d \leq 2$, then $X_t \xrightarrow{d} 0$ as $t \to \infty$.*

As we know, the general form of branching mechanism of superprocesses is that

$$\psi(x,z) = b(x)z + c(x)z + \int_0^\infty (e^{-uz} - 1 + uz)n(x,\mathrm{d}u), x \in R^d, \quad z > 0, \tag{3}$$

where $n(x, \mathrm{d}u)$ is a kernel from R^d to $(0,\infty)$ and $b(x)$, $c(x) > 0$ and $\int_0^\infty (u \wedge u^2)n(x,\mathrm{d}u) > 0$ are all bounded Borel measurable functions on R^d. For this, we have

Corollary 1.3 *Under the conditions of Theorem 1.1 with $\psi(x,z)$ given by (3). If $\psi(x,z) \geq \gamma z^{1+\beta}$ and $\beta d \leq 2$, then $X_t \xrightarrow{d} 0$ as $t \to \infty$.*

Remark 1 Theorem 1.1 and 1.2 extend the results in [3], while the later only discuss the special case $\beta \equiv 1$. This note reveals in some sense that the joint influence of dimension of the underlying space and branching mechanism on the properties of superprocesses.

Remark 2 The significance of Corollary 1.3 can be seen from a simple example, e.g., if

$\psi(x,z)=\gamma_1 z^{1+\beta_1}+\gamma_2 z^{1+\beta_2}$ and $\beta_1\neq\beta_2$, then $\psi(x,z)$ can't be expressed as the form of $\gamma z^{1+\beta}$.

2. Proof of main results

In order to prove the main results, we first give two Lemmas.

Lemma 2.1[3,5] *If $p>d$, then $\lambda(\mathrm{d}x)\in M_p(R^d)$, and $\mu(\mathrm{d}x)=m(x)\mathrm{d}x=m(x)\lambda(\mathrm{d}x)\in M_p(R^d)$.*

Lemma 2.2 *Let $p(t,x,y)$ be the transition probability density function of ξ_t, then for arbitrary $d\geq 1$, we have*

$$\int_{|x|>(t\ln t)^{1/2}} p(t,x,0)\mu(\mathrm{d}x)\longrightarrow 0 \quad as\ \ t\to\infty. \tag{4}$$

Proof We prove the Lemma by the method taking from [8]. Noting the hypothesis (H1), it is sufficient to prove that $\int_{\sqrt{t\ln t}}^{\infty} t^{-\frac{d}{2}}e^{-\alpha\frac{r^2}{t}}\bar{\mu}(\mathrm{d}r)\longrightarrow 0$ as $t\to\infty$, where $\bar{\mu}(r,r+a)$ denotes the $\mu(\mathrm{d}x)$-measure of the annulus with inner radius r and outer radius $r+a$. Since for sufficient large x, $e^{-x^2}<c_1(d)x^{-d-1}$ (here and later, we use c to denote different constants, and $c(\cdot)$ to denote the constant depending on $\cdot$). This implies that

$$\begin{aligned}\int_{\sqrt{t\ln t}}^{\infty} t^{-\frac{d}{2}}e^{-\alpha\frac{r^2}{t}}\bar{\mu}(\mathrm{d}r) &\leq c_2(d,\alpha)\int_{\sqrt{t\ln t}}^{\infty}\sqrt{t}/r^{d+1}\bar{\mu}(\mathrm{d}r)\\ &\leq c_2(d,\alpha)\sum_{k=K_t}^{\infty}(\sqrt{t}/2^{k(d+1)})\bar{\mu}([2^k,2^{k+1}]),\end{aligned}$$

for sufficient large t, where $K_t=[\log_2(\sqrt{t\ln t})]$, and $[\cdot]$ denotes "greatest integer in". From the hypothesis (H2), we know that $\bar{\mu}([2^k,2^{k+1}])\leq c_3(d,P)2^{kd}$, where P is the constant in (H2). Therefore, $c_2(d,\alpha)\int_{\sqrt{t\ln t}}^{\infty}\sqrt{t}/r^{d+1}\bar{\mu}(\mathrm{d}r)\leq c_4(d,\alpha,P)/\sqrt{\ln t}$ holds for sufficient large t. This implies the conclusion.

Proof of Theorem 1.1 For $\forall f\in pC_c(R^d)$, the Laplace functional of X_t reads as $E^{\mu}\exp(-\langle f,X_t\rangle)=\exp(-\langle V_t,\mu\rangle)$, where

$$V_t=P_tf-\gamma\int_0^t P_{t-s}(V_s^{1+\beta})\mathrm{d}s, \tag{5}$$

in which P_t is the semigroup of ξ_t. Integrating equation (5) with the invariant measure μ, we conclude that

$$\langle V_t,\mu\rangle=\langle f,\mu\rangle-\gamma\int_0^t\langle V_s^{1+\beta},\mu\rangle\mathrm{d}s. \tag{6}$$

As $t\to\infty$, from (6) we know that $\langle V_t,\mu\rangle$ decreases with lower bound 0, therefore $V_\infty(f)=\lim_{t\to\infty}\langle V_t,\mu\rangle$ exists. Thus in order to prove the result, it is sufficient to prove that $V_\infty(f)\neq 0$ for some $f\in pC_c(R^d)$. In particular, it is sufficient to prove the conclusion for some indication function of some compact set in R^d.

Formula (5) implies that $0 \le V_t \le P_t f$. By equation (6) we have

$$\langle V_t, \mu\rangle \ge \langle f, \mu\rangle - \gamma \int_0^t \langle P_{t-s}(P_s(f))^{1+\beta}, \mu\rangle \mathrm{d}s. \tag{7}$$

Let $f(x) = l1_A(x)$, where $1_A(x)$ is the indication function of A, and $A = \{x | x = (x_1, x_2, \cdots, x_d) \in R^d, x_1^2 + x_2^2 + \cdots + x_d^2 \le k\}$ is a closed ball in R^d. From Jensen inequality we obtain $(P_s f(x))^{1+\beta} = (E_x f(\xi_s))^{1+\beta} \le E_x f^{1+\beta}(\xi_s) = P_s f^{1+\beta}(x)$. From this inequality and condition (H1), we have

$$\begin{aligned}\int_0^1 \langle V_s^{1+\beta}, \mu\rangle \mathrm{d}s &\le \int_0^1 \langle (P_s f)^{1+\beta}, \mu\rangle \mathrm{d}s \\ &\le \int_0^1 \mathrm{d}s \int_{R^d} \mu(\mathrm{d}x) P_s f^{1+\beta}(x) \\ &= l^{1+\beta} \int_0^1 \mathrm{d}s \int_{R^d} \mu(\mathrm{d}x) \int_{R^d} 1_A(y) p(s, x, y) \mathrm{d}y \\ &\le c_5(d, M, P) l^{1+\beta} \int_0^1 \mathrm{d}s \int_{R^d} 1_A(y) \mathrm{d}y \\ &= \frac{c_5(d, M, P) \pi^{\frac{d}{2}} k^d}{\Gamma(\frac{d+2}{2})} l^{1+\beta} \\ &= c_6(d, M, P, k) l^{1+\beta},\end{aligned} \tag{8}$$

where $\frac{\pi^{\frac{d}{2}} k^d}{\Gamma(\frac{d+2}{2})}$ is the volume of the ball A. Furthermore, under the condition of (H1) , by Jensen inequality and $\beta d > 2$, we find that

$$\begin{aligned}\int_1^t \langle V_s^{1+\beta}, \mu\rangle \mathrm{d}s &\le \int_1^t \mathrm{d}s \int_{R^d} \mu(\mathrm{d}x) \Big(\int_{R^d} M s^{-d/2} e^{-\frac{\alpha}{s}|y-x|^2} l 1_A(y) \mathrm{d}y\Big)^{1+\beta} \\ &\le \int_1^t \mathrm{d}s \int_{R^d} \mu(\mathrm{d}x) \int_{R^d} M^{1+\beta} l^{1+\beta} 1_A(y) s^{-\frac{d}{2} - \frac{\beta d}{2}} e^{-\frac{\alpha(1+\beta)}{s}|y-x|^2} \mathrm{d}y \\ &\le c_7(\alpha, M, P) l^{1+\beta} \int_1^t \mathrm{d}s \int_{R^d} 1_A(y) s^{-\frac{\beta d}{2}} \mathrm{d}y \\ &\le c_8(d, \alpha, M, P, k) \frac{2}{\beta d - 2} (1 - t^{1-\frac{\beta d}{2}}) l^{1+\beta} \\ &\le c_9(d, \alpha, M, P, k, \beta) l^{1+\beta}.\end{aligned} \tag{9}$$

From equation (7), (8) and (9), we obtain

$$\begin{aligned}V_\infty(f) &\ge \langle f, \mu\rangle - \gamma \lim_{t\to\infty} \int_0^t \langle V_s^{1+\beta}, \mu\rangle \mathrm{d}s \\ &= l\langle 1_A, \mu\rangle - \gamma \int_0^1 \langle V_s^{1+\beta}, \mu\rangle \mathrm{d}s - \gamma \lim_{t\to\infty} \int_1^t \langle V_s^{1+\beta}, \mu\rangle \mathrm{d}s \\ &\ge l\mu(A) - l^{1+\beta} \gamma (c_6 + c_9).\end{aligned}$$

It is not difficult to see that if we choose $l < (\frac{\mu(A)}{\gamma(c_6+c_9)})^{\frac{1}{\beta}}$, then $V_\infty(f) > 0$. This completes the proof.

Proof of Theorem 1.2 For $\forall f \in pC_c(R^d)$, the Laplace functional of X_t can be expressed as $E^\mu \exp(-\langle f, X_t\rangle) = \exp(-\langle V_{t,}\mu\rangle)$, where V_t is the solution of the equation $V_t = P_t f - \int_0^t P_{t-s}(\psi(V_s))\mathrm{d}s$. From [9] we know that, in order to prove this Theorem, it's sufficient to show that $\lim_{t\to\infty}\langle V_t, \mu\rangle = 0$.

Set $S_t := \{x \in R^d, |x| \le ((k+t)\ln(k+t))^{1/2}\}$ with k being a positive constant. From [3] and Lemma 2.1 and 2.2 we know that $\int_{S_t^c} V_t(x)\mu(\mathrm{d}x) \longrightarrow 0$ as $t \to \infty$. It remains to prove $\int_{S_t} V_t(x)\mu(\mathrm{d}x) \longrightarrow 0$ as $t \to \infty$. For the sake, we introduce the function $W(t)$, which is the solution of the following equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}W(t) = -\frac{\gamma}{P}W^{1+\beta}(t) + \frac{h'(t)}{h(t)}W(t) + g(t)h(t), \\ W(0) = \frac{1}{\lambda(S_0)}\int_{S_0} f(x)\mu(\mathrm{d}x). \end{cases} \tag{10}$$

Here P is the constant in condition (H) and

$$h(t) = \frac{1}{\lambda(S_t)} = c_{10}\cdot((k+t)\ln(k+t))^{-d/2}, \quad g(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t^c} V_t(x)\mu(\mathrm{d}x).$$

It is easy to show that $\int_0^\infty g(t)dt = \infty$. Now putting

$$Z(t) = \frac{1}{(W(t)\lambda(S_t))^\beta}, \quad Z(0) = \frac{1}{(W(0)\lambda(S_0))^\beta}. \tag{11}$$

From (10) and (11) we see that $\frac{\mathrm{d}Z(t)}{\mathrm{d}t} = \frac{\beta\gamma}{P}h(t)^\beta - \beta g(t)Z(t)^{1+\frac{1}{\beta}}$, i.e.,

$$Z(t) = Z(0) + \frac{\beta\gamma}{P}\int_0^t h(s)^\beta \mathrm{d}s - \beta\int_0^t g(s)Z(s)^{1+\frac{1}{\beta}}\mathrm{d}s. \tag{12}$$

Noting that $\int_0^\infty h(t)^\beta \mathrm{d}t = c_{10}\int_0^\infty((K+t)\ln(k+t))^{-\beta d/2}\mathrm{d}t = \infty$ holds if and only if $\beta d \le 2$. If $\sup_{t>0} Z(t) < \infty$, then the right side of (12) would be infinity and the left hand would not be so. This is absurd, therefore $Z(t) \stackrel{t\to\infty}{\longrightarrow} \infty$. It follows that $W(t)\lambda(S_t) \stackrel{t\to\infty}{\longrightarrow} 0$. Now we aim to show that

$$\int_{S^t} V_t(x)\mu(\mathrm{d}x) \le W(t)\lambda(S_t).$$

Noting that when $t = 0$, the above inequality is strictly equal, it is sufficient to show that if $\int_{S_{t_0}} V_{t_0}(x)\mu(\mathrm{d}x) = W(t_0)\lambda(S_{t_0})$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t} V_t(x)\mu(\mathrm{d}x)|_{t=t_0} \le \frac{\mathrm{d}}{\mathrm{d}t}(W(t)\lambda(S_t))|_{t=t_0}.$$

First of all, we have $\frac{\mathrm{d}}{\mathrm{d}t}W(t)\lambda(S_t) = -\frac{\gamma W(t)^{1+\beta}}{Ph(t)} + g(t)$.

Next, noting that $\mu(\mathrm{d}x)$ is an invariant measure of ξ_t, therefore $\int_{R^d} V_t(x)\mu(\mathrm{d}x) = \langle f, \mu\rangle - \gamma\int_0^t\langle V_s^{1+\beta}, \mu\rangle\mathrm{d}s$. It follows that $\frac{\mathrm{d}}{\mathrm{d}t}(\int_{S_t} V_t(x)\mu(\mathrm{d}x) + \int_{S_0^t} V_t(x)\mu(\mathrm{d}x)) = -\gamma\langle V_t^{1+\beta}, \mu\rangle$. That is $\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t} V_t(x)\mu(\mathrm{d}x) = g(t) - \gamma\langle V_t^{1+\beta}, \mu\rangle \le g(t) - \gamma\int_{S_t} V_t^{1+\beta}(x)\mu(\mathrm{d}x)$. Thanks of Hölder inequality, $|m(x)| \le P$ and $P \ge 1$, we arrive at

$$\int_{S_t} V_t^{1+\beta}(x)\mu(\mathrm{d}x) \ge \frac{1}{P\lambda(S_t)^\beta}(\int_{S_t} V_t(x)\mu(\mathrm{d}x))^{1+\beta}.$$

Finally, it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t} V_t(x)\mu(\mathrm{d}x)|_{t=t_0} \le g(t_0) - \frac{\gamma}{P}W(t_0)^{1+\beta}\lambda(S_{t_0}) = \frac{\mathrm{d}}{\mathrm{d}t}(W(t)\lambda(S_{t_0}))|_{t=t_0}.$$

This completes the proof.

Proof of Corollary 1.3 In equation (2), it is ease to find that if $\psi(\cdot,\cdot)$ becomes bigger, then the solution will become smaller. From this fact we know that the Corollary holds.

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一类超一致椭圆扩散过程的极限定理

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摘 要: 本文研究了分支特征为 $\psi(x,z)=\gamma z^{1+\beta}(0<\beta\le 1)$ 形式的超一致椭圆扩散过程，当初始值 $X_0(dx)$ 为底过程的某类不变测度时，给出了当空间维数 d 满足 $\beta d\le 2$ 时，超过程 X_t 依分布收敛于 0 测度，当 $\beta d>2$ 时， X_t 则依分布收敛于一个非退化的随机测度.