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A Global Existence Theorem for Semilinear Wave Equations in Five Space Dimension *

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Abstract: This paper concerns the global existence of solutions to the semi-linear wave equation $u_{tt} - \Delta u = G(u)$ in five space dimensions, where $G(u) \sim |u|^p$ with $p > \frac{3+\sqrt{17}}{4}$. We used the classical iteration method and technique estimates to show that a classical global solution exists for the radially symmetric equations with small and compact supported initial data.

Key words: global existence; semi-linear wave equation.

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1. Introduction

We are concerned with the following Cauchy problem for nonlinear wave equations

$$\begin{cases} u_{tt} - \Delta u = G(u), & x \in \mathbf{R}^n, t \in \mathbf{R}, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \end{cases} \tag{1}$$

where $G(u) \sim |u|^p$ with $p > 1$, $f(x)$ and $g(x)$ are smooth and compact supported functions. Since large data lead to blow up of the solution to this type of equations, a natural question is the following: find a critical value p_0 such that the equation has global solutions for sufficiently small initial data with $p > p_0$, and most solutions blow up in finite time with $p < p_0$.

John[5] has proved that for $n = 3$ and $1 < p < 1 + \sqrt{2}$, a global solution does not exist for any smooth non-trivial data with compact support; and for $p > 1 + \sqrt{2}$, a global solution exists provided that the initial data are sufficiently small. Glassey[3] has showed that for $n = 2$ the solution must blow up in finite time when $1 < p < \frac{3+\sqrt{17}}{2}$, and a small global solution exists when $p > \frac{3+\sqrt{17}}{2}$ in [4]. Moreover, Glassey[4] conjectured that for $n \geq 2$, there exists a critical exponent $p_0(n)$ such that most solutions blow up in finite time when $1 < p < p_0(n)$, and a small solution exists for $p > p_0(n)$ where $p_0(n)$ is the

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positive root of quadratic equation $(n-1)p^2-(n+1)p-2=0$. The work of John and Glassey concluded that $p_0(2)=\frac{3+\sqrt{17}}{2}$ and $p_0(3)=1+\sqrt{2}$. Sideris[12] has proved that for $n\geq 4$ and $1<p<p_0(n)$, the solutions blow up in finite time. In a recent important work of Georgiev, Lindblad, and Sogge[1,7], they showed that when $p>p_0(n)$ with $n\geq 3$, the global(weak) solution exists for sufficiently small initial data in some mixed norm spaces. Their argument relies on a generalized Strihartz estimate. Namely the conjecture is proved for for arbitrary small initial data in a weak sense. There are also some results for $p=p_0(n)$. Schaeffer[10] has proved that when $p=p_0(2)=\frac{3+\sqrt{17}}{2}$ for $n=2$, and when $p=p_0(3)=1+\sqrt{2}$ for $n=3$, the solutions blow up in finite time.

In this paper, we consider the radially symmetric version of Equation (1)

$$\begin{aligned} &u_{tt}-\tfrac{n-1}{r}u_r-u_{rr}=G(u), \quad r\geq 0,\ t\geq 0,\\ &u(r,0)=f(r), \quad u_t(r,0)=g(r), \end{aligned} \tag{2}$$

in five space dimensions with $f(r)$, $g(r)$ compact supported and G depending only on u. The main theorem is stated as the following

Theorem 1 *Assume that f and g are even functions, $f,g\in C_0^2(\mathbf{R})$, and $\operatorname{supp}\{f,g\}\subset\{r:|r|\leq k\}$. If G satisfies*

$$|G(u)|\leq A|u|^p, \qquad |\frac{\partial G(u)}{\partial u}|\leq B|u|^{p-1}, \qquad A,B>0,$$

then Equation (2) with $n=5$ has a unique global solution in the space $\Omega_{p,k}$ provided that $p>p_0(5)=\frac{3+\sqrt{17}}{4}$ and $\sum_{i=0}^{2}(||f^{(i)}||_\infty+||g^{(i)}||_\infty)$ is sufficiently small, where

$$\Omega_{p,k}=\begin{cases} u\in C^0(\mathrm{R}\times[0,\infty))\mid \mathbf{ru},(\mathbf{ru})_\mathbf{r}\in \mathbf{C^0}(\mathbf{R}\times[0,\infty)),\\ u \ \ \textit{is even and } u(r,t)=0 \ \textit{if}\ |r|>t+k,\\ ||u||=\sup(|\varphi u|+|\varphi(ru)|+|\varphi(ru)_r|)<+\infty,\\ \varphi(r,t)=(r+t+2k)(t-r+2k)^{2p-3}. \end{cases}$$

As in many other global existence theory, the factor φ in the definition of $\Omega_{k,p}$ takes into the account of the decay properties of the free wave equation. Important thing here is to get a right exponent to work with. Our space is easy to describe since it involves only supremum norm, and the method is totally different from what Lindbald and Sogge used. The argument is based simply contraction mapping principle.

Recall that the solution to the free equation[8] ($G=0$) is given by

$$u^0(r,t)=C_0r^{-3}\int_{|r-t|}^{r+t}\lambda(r^2+\lambda^2-t^2)g(\lambda)\mathrm{d}\lambda+\frac{\partial}{\partial t}\left[C_0r^{-3}\int_{|r-t|}^{r+t}\lambda(r^2+\lambda^2-t^2)f(\lambda)\mathrm{d}\lambda\right]. \tag{3}$$

By Duhamel Principle, solving Equation (2) is equivalent to finding a fixed point for the operator T in a suitable space, where

$$Tu(r,t)=u^0(r,t)+T_1u(r,t), \tag{4}$$

where

$$T_1 u(r,t) = \frac{C_0}{r^3} T_0 u(r,t),$$
$$T_0 u(r,t) = \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda(r^2+\lambda^2-(t-\tau)^2) G(u(\lambda,\tau)) \mathrm{d}\lambda \mathrm{d}\tau. \tag{5}$$

The goal in this paper is to prove that T is a contraction map in $\Omega_{k,p}$. Therefore We need to prove: (a) $u^0 \in \Omega_{k,p}$ if f and g are smooth enough; (b) T_0 is a well-defined map from $\Omega_{k,p}$ to itself; and (c) T_0 is actually a contracting map in a small neighborhood of $0 \in \Omega_{k,p}$. In Section 2, we will prove part (a) and state some technical lemmas for the proof of part (b). In Section 3, we will concentrate on the proof of part (b). In Section 5, part (c) and some technical lemma will be proved.

2. Free solution and some basic estimates

In this part we will see that $u^0 \in \Omega_{k,p}$ and state some technical lemmas to be used in Section 2.

Lemma 1 *Let $h \in C^2(\mathbf{R})$ be an even function with $h(r) = 0$ for $|r| \geq k$. Define*

$$v(r,t) = \int_{r-t}^{r+t} \lambda(r^2+\lambda^2-t^2) h(\lambda) \mathrm{d}\lambda$$

and $||v||_\infty = \sup\{|v(r,t)|; r \in \mathbf{R}, t \geq 0\}$ *for* $v \in C^0(\mathbf{R}\times[0,\infty))$. *Then there exists a constant A_1 such that*

$$||wr^{-3}v||_\infty + ||wr^{-3}v_t||_\infty + ||wr^{-2}v||_\infty + ||wr^{-2}v_t||_\infty + ||wr^{-2}v_r||_\infty + ||wr^{-2}v_{rt}||_\infty$$
$$\leq A_1(||h||_\infty + ||h'||_\infty + ||h''||_\infty),$$

where $w(r,t) = t + r + 2k$.

Proof Without loss of generality, we assume $r > 0$, $t > 0$ and note that $v(r,t) \neq 0$ only when $t-k < r < t+k$. First we will estimate $r^{-1}v_r$. Differentiate v with respect to r

$$v_r = \int_{r-t}^{r+t} 2\lambda r h(\lambda) d\lambda + 2r(r+t)^2 h(r+t) - 2r(r-t)^2 h(r-t) = 2rl(r,t),$$

where $l(r,t) = \int_{r-t}^{r+t} \lambda h(\lambda)\mathrm{d}\lambda + (r+t)^2 h(r+t) - (r-t)^2 h(r-t)$. Since h is even and vanishes outside $(-k,k)$, we have the following

$$|v_r(r,t)| \leq 2r\left[\int_{r-t}^{r+t} |\lambda h(\lambda)|\mathrm{d}\lambda + (r-t)^2|h(r+t)| + (r-t)^2|h(r-t)|\right]$$
$$\leq 2r[2k^2||h||_\infty + 2k^2||h||_\infty] = 8k^2 r||h||_\infty,$$

hence

$$|r^{-1}v_r(r,t)| \leq 8k^2||h||_\infty. \tag{6}$$

Differentiating $l(r,t)$ with respect to r yields

$$l_r(r,t) = 3(r+t)h(r+t) - 3(r-t)h(r-t) + (r+t)^2h'(r+t) - (r-t)^2h'(r-t),$$

hence $|l_r(r,t)| \le 6k||h||_\infty + 2k^2||h'||_\infty$ and $|l(r,t)| = |\int_0^r l_r(s,t)\mathrm{d}s| \le (6k||h||_\infty + 2k^2||h'||_\infty)r$. Therefore $|v_r(r,t)| = 2r|l(r,t)| \le r^2(12k||h||_\infty + 4k^2||h||_\infty)$, i.e.,

$$|r^{-2}v_r(r,t)| \le 12k||h||_\infty + 4k^2||h'||_\infty. \tag{7}$$

Now let us estimate $|wr^{-2}v_r|$. If $t-r+2k < \frac{1}{2}(r+t+2k)$, i.e., $r > \frac{r+t+2k}{4}$, (6) implies

$$\left|\frac{r+t+2k}{r^2}v_r(r,t)\right| \le \frac{4}{r}|v_r(r,t)| \le 4|r^{-1}v_r(r,t)| \le 32k^2||h||_\infty.$$

If $t-r+2k > \frac{1}{2}(t+r+2k)$, by $-k < t-r < k$ and (6)

$$\left|\frac{t+r+2k}{r^2}v_r(r,t)\right| \le \left|\frac{2(t-r+2k)}{r^2}v_r(r,t)\right| \le 6k\left|\frac{1}{r^2}v_r(r,t)\right| \le 6k(12k||h||_\infty + 4k^2||h'||_\infty).$$

In order to estimate $|wr^{-3}v|$, we need to estimate $|r^{-3}v|$ and $|r^{-2}v|$. Since $|v_r(r,t)| \le r^2(12k||h||_\infty + 4k^2||h'||_\infty)$ and $v(0,t)=0$, $|v(r,t)| \le \int_0^r |v_r(\lambda,t)d\lambda \le (12k||h||_\infty + 4k^2||h||_\infty)r^3$, i.e.,

$$|r^{-3}v(r,t)| \le 12k||h||_\infty + 4k^2||h'||_\infty.$$

If $r > k$,

$$|v(r,t)| \le \int_{r-t}^{r+t} |\lambda(r^2+\lambda^2-t^2)h(\lambda)|\mathrm{d}\lambda \le ||h||_\infty \int_{-k}^{k} |\lambda|(k^2+|r-t|\cdot|r+t|)\mathrm{d}\lambda \le 8k^3r||h||_\infty.$$

So $|r^{-1}v(r,t)| < 8k^3||h||_\infty$, therefore $|r^{-2}v(r,t)| \le ||\frac{1}{k}r^{-1}v(r,t)| \le 8k^2||h||_\infty$.

If $r \le k$, $|r^{-2}v(r,t)| \le k|r^{-3}v(r,t)| \le 12k^2||h||_\infty + 4k^3||h'||_\infty$. Therefore

$$|r^{-1}v(r,t)| \le k|r^{-2}v(r,t) \le 12k^3||h||_\infty + 4k^3||h'||_\infty.$$

Let's estimate $|wr^{-3}v|$. If $t-r+2k > \frac{1}{2}(t+r+2k)$ and note that $|t-r| < k$, we have

$$|\frac{w}{r^3}v(r,t)| \le |\frac{2(r+t+2k)}{r^3}v(r,t)| \le 6k|\frac{1}{r^3}v(r,t)| \le 6k(12k||h||_\infty + 4k^2||h'||_\infty)$$

If $t-r+2k \le \frac{1}{2}(r+t+2k)$, i.e., $r > \frac{r+t+2k}{4}$, then

$$|\frac{w}{r^3}v(r,t)| \le |\frac{4}{r^2}v(r,t)| \le 4(12k^2||h||_\infty + 4k^3||h||_\infty).$$

Therefore $|\frac{w}{r^3}v(r,t)| \le 72k^2||h||_\infty + 24k^3||h'||_\infty$. Using similar arguments, we have

$$\left|\frac{w}{r^2}v(r,t)\right| \le 72k^3||h||_\infty + 24k^3||h'||_\infty.$$

To estimate $|wr^{-2}v_{rt}|$, we need to estimate $|r^{-1}v_{rt}|$ and $|r^{-2}v_{rt}|$.

$$v_t = \int_{r-t}^{r+t} -2\lambda t h(\lambda) d\lambda + 2r(r+t)^2 h(r+t) + 2r(r-t)^2 h(r-t),$$
$$v_{rt} = 6r(r+t)h(r+t) + 6r(r-t)h(r-t) + 2r(r+t)^2 h'(r+t) + 2r(r-t)^2 h'(r-t).$$

Since h is even and vanishes outside $(-k,k)$, we have

$$|r^{-1}v_{rt}(r,t)| \le 12k||h||_\infty + 4k^2||h'||_\infty \quad \text{and} \quad \lim_{r\to 0} r^{-1}v_{r,t} = 0,$$
$$\begin{aligned} |\frac{\partial}{\partial r}(r^{-1}v_{rt}(r,t))| &= |6h(r+t) + 6h(r-t) + 10(r+t)h'(r+t) + \\ &\quad 10(r-t)h'(r-t) + 2(r+t)^2 h''(r+t) + 2(r-t)^2 h''(r-t)| \\ &\le 12||h||_\infty + 20k||h'||_\infty + 2k^2||h''||_\infty, \\ |r^{-1}v_{rt}(r,t)| &\le \int_0^r |\frac{\partial}{\partial z}(z^{-1}v_{rt}(z,t)|\mathrm{d}z \\ &\le (12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty)r. \end{aligned}$$

Therefore

$$|r^{-2}v_{rt}(r,t)| \le 12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty. \tag{8}$$

A similar argument as before yields $|wr^{-2}v_{rt}(r,t)| \le 6k(12||h||_\infty + 20k||h''||_\infty + 4k^2||h''||_\infty)$. Since $v_t(0,t) = 0$, and by (8) we have $|v_t(r,t)| \le \int_0^r |v_{rt}(s,t)|\mathrm{d}s \le (12||h||_\infty + 20||h'||_\infty + 4k^2||h''||_\infty)r^3$, i.e.,

$$|r^{-3}v_t(r,t)| \le 12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty.$$

On the other hand $|v_t| \le \int_{r-t}^{r+t} |2sth(s)|\mathrm{d}s + 2r(r+t)^2|h(r+t)| + 2r(r-t)^2|h(r-t)|$.

If $r > k$, then

$$|v_t(r,t)| \le \int_{-k}^{k} 2t|s|h(s)|\mathrm{d}s + 4rk^2||h||_\infty \le 4tk^2||h||_\infty + 4rk^2||h||_\infty \le 12rk^2||h||_\infty$$

That is $|r^{-1}v_t(r,t)| \le 12k^2||h||_\infty, |r^{-2}v_t(r,t)| \le 12k||h||_\infty$, for $r > k$.

If $r \le k$,

$$|r^{-1}v_t(r,t)| \le k^2(12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty),$$
$$|r^{-2}v_t(r,t)| \le k(12||h||_\infty + 20k||h'||_\infty + 4k^2||h''||_\infty).$$

By a similar argument as before, there exists a constant $C(k)$ such taht

$$|wr^{-2}v_t| \le C(k)(||h||_\infty + ||h'||_\infty + ||h''||_\infty), \ |wr^{-3}v_t| \le C(k)(||h||_\infty + ||h'||_\infty + ||h''||_\infty).$$

Lemma 2 *There exists a constant A_2 such that*

$$||u^0|| \le A_2(||f||_\infty + ||f'||_\infty + ||f''||_\infty + ||g||_\infty + ||g'||_\infty + ||g''||_\infty).$$

Proof Recall $||u^0|| = \sup\{|\varphi u^0| + |\varphi(ru^0)| + |\varphi(ru^0)_r|\}$ and note $t-r+2k \le 3k$ on the support of u^0, then

$$|\varphi(r,t)u^0(r,t)| \le (3k)^{2p-3}\left|w(r,t)r^{-3}\left(v^g(r,t)+v_t^f(r,t)\right)\right|$$
$$|\varphi(r,t)ru^0(r,t)| \le (3k)^{2p-3}\left|w(r,t)r^{-2}\left(v^g(r,t)+v_t^f(r,t)\right)\right|$$
$$|\varphi(r,t)(ru^0)_r(r,t)| \le (3k)^{2p-3}|w(r,t)(-2r^{-3}v^g(r,t)+r^{-2}v_r^g(r,t)-2r^{-3}v_t^f(r,t)+r^{-2}v_{rt}^f(r,t))|,$$

where $v^g = \int_{r-t}^{r+t}\lambda(r^2+\lambda^2-t^2)g(\lambda)\mathrm{d}\lambda$, $v^f = \int_{r-t}^{r+t}\lambda(r^2+\lambda^2-t^2)f(\lambda)\mathrm{d}\lambda$.

Therefore, the lemma follows from Lemma 1.

In order to see that T is a well-defined operator on $\Omega_{k,p}$, we need the following technical lemmas. We first state them here and will give the proof in Section 5 since their proof is purely analysis and has nothing to do with PDEs.

Lemma 3 *For $p \in (\frac{3+\sqrt{17}}{4}, 2)$ and $0 < r < t+k$, let $t_1 = \max\{0, \frac{1}{2}(t-r-k)\}$ and $t_2 = \max\{0, \frac{1}{2}(t+r-k)\}$, there exists a constant C_1 such that*

$$\int_{t_1}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=|r-t+\tau|}\mathrm{d}\tau \le \frac{C_1}{(t-r+2k)^{2p-2}},\quad \int_{t_2}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}\mathrm{d}\tau \le \frac{C_1}{(t-r+2k)^{2p-2}},$$

where $z(r) = \min\{1, \frac{1}{r}\}$.

Lemma 4 *For $p \in (\frac{3+\sqrt{17}}{4}, 2)$, there exists a constant C_2 such that for $0 < r < t+k$,*

$$\int_{t_1}^{t}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=|r-t+\tau|}\mathrm{d}\tau \le \frac{C_2}{(t-r+2k)^{2p-2}},\quad \int_{t_2}^{t}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}\mathrm{d}\tau \le \frac{C_2}{(t-r+2k)^{2p-2}}.$$

3. Properties of the Operator T_1

In this section we will establish the following

Theorem 2 *The map T_1 is a well-defined map on $\Omega_{k,p}$. Furthermore*

$$||T_1u|| \le C||u||^p.$$

To prove this theorem, we need to establish: (i) $Tu(r,t)$ is even in r; (ii) $Tu(r,t)=0$ for $|r| \ge t+r$; (iii) Tu, rTu and $(rTu)_r$ are continuous; (iv) $||Tu|| < \infty$.

Since $\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau))$ is an odd function in λ, therefore

$$\begin{aligned} T_0u(r,t) &= \int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau))\mathrm{d}\lambda \\ &= \int_0^t\int_{r-t+\tau}^{r+t-\tau}\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau))\mathrm{d}\lambda. \end{aligned}$$

Hence $\frac{1}{r^3}T_0u(r,t) = \frac{1}{(-r)^3}T_0u(-r,t)$. i.e., $T_1u(r,t) = T_1u(-r,t)$.

From the expressions of T_0u, we can easily see that $Tu(r,t)=0$ for $|r|\ge t+k$ when $u\in\Omega_{k,p}$. It remains to prove (iii) and (iv).

For $u\in\Omega_{p,k}$ and $0<r<t+k$, we have $|u(r,t)|\le\frac{\|u\|}{\varphi(r,t)}$, $|u(r,t)|\le\frac{\|u\|}{r\varphi(r,t)}$. Hence

$$|u(r,t)|\le\frac{\|u\|z(r)}{\varphi(r,t)}\quad\text{and}\quad|u_r(r,t)|\le\frac{|(ru)_r|+|u|}{r}\le\frac{2\|u\|}{r\varphi(r,t)}.$$

In order to prove (iii) and (iv), Several lemmas are needed.

Lemma 5 *There exists a constant C_3 such that for $p\in(\frac{3+\sqrt{17}}{4},2)$*

$$\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\lambda G(u(\lambda,\tau))\mathrm{d}\lambda\mathrm{d}\tau\le\frac{C_3\|u\|^p}{(t-r+2k)^{2p-2}}\ .$$

Proof Since $G\sim|u|^p$, there exists an $A>0$ such that $G\le A|u|^p$, we have

$$\lambda G(u(\lambda,\tau))\le\lambda A|u(\lambda,\tau)|^p\le\frac{A\lambda z^p(\lambda)\|u\|^p}{\varphi^p(\lambda,\tau)}\le\frac{A\|u\|^p\lambda^{1-p}}{\varphi^p(\lambda,\tau)},$$

$$\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\lambda G(u(\lambda,\tau))\mathrm{d}\lambda\mathrm{d}\tau\le A\|u\|^p\int_0^t\int_{|r-t+\tau|}^{t+r-\tau}\frac{\lambda^{1-p}}{\varphi^p(\lambda,\tau)}\mathrm{d}\lambda\mathrm{d}\tau.$$

Let $\tau+\lambda=\beta$ and $\tau-\lambda=\alpha$, then

$$\begin{aligned}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\lambda G(u(\lambda,\tau))\mathrm{d}\lambda\mathrm{d}\tau&\le A\|u\|^p\int_{-k}^{t-r}\int_{t-r}^{t+r}\frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\alpha+2k)^{p(2p-3)}(\beta+2k)^p}\frac{1}{2}\mathrm{d}\beta\mathrm{d}\alpha\\
&=A\|u\|^p[\int_{-k}^{\frac{t-r}{2}}(\alpha+2k)^{-p(2p-3)}d\alpha\int_{t-r}^{t+r}\frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}\mathrm{d}\beta+\\
&\int_{\frac{t-r}{2}}^{t-r}(\alpha+2k)^{-p(2p-3)}\mathrm{d}\alpha\int_{t-r}^{t+r}\frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}\mathrm{d}\beta].\end{aligned}\tag{9}$$

Now we estimate the following integral for $1-p-l<-2$.

Case 1 if $t-r-\alpha\le 2$,

$$\begin{aligned}\int_{t-r}^{t+r}\frac{(\frac{\beta-\alpha}{2})^{1-p}}{(\beta+2k)^l}\frac{1}{2}\mathrm{d}\beta&=\int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}}\frac{\lambda^{1-p}}{(2\lambda+\alpha+2k)^l}\mathrm{d}\lambda\le\int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}}\frac{\lambda^{1-p}}{(2\lambda+k)^l}\mathrm{d}\lambda\le\int_0^\infty\frac{\lambda^{1-p}}{(2\lambda+k)^l}\\
&\le\frac{1}{k^l}\int_0^1\lambda^{1-p}\mathrm{d}\lambda+\int_1^\infty\frac{\lambda^{1-p}}{(2\lambda+k)^l}\mathrm{d}\lambda=\frac{1}{k^l}\frac{1}{2-p}+\frac{1}{2^l}\frac{1}{p+l-2}\\
&\le\left(\frac{1}{k^l}\frac{1}{2-p}+\frac{1}{2^l}\frac{1}{p+l-2}\right)\frac{(2+k)^{p+l-2}}{(t-r-\alpha+k)^{p+l-2}},\end{aligned}$$

where we used the fact $k<t-r-\alpha+k\le 2+k$.

Case 2 If $t-r-\alpha>2$,

$$\int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}}\frac{\lambda^{1-p}}{(2\lambda+k)^l}\mathrm{d}\lambda\le\frac{1}{2^l}\int_{\frac{t-r-\alpha}{2}}^{\infty}\lambda^{1-p-l}\mathrm{d}\lambda=\frac{1}{2^l(l+p-2)}\left(\frac{t-r-\alpha}{2}\right)^{2-l-p}.$$

Since $t-r-\alpha>2$, we have $\frac{k}{2}(t-r-\alpha)>k$. Therefore $(1+\frac{k}{2})(t-r-\alpha)>t-r-\alpha+k$, i.e., $t-r-\alpha>\frac{2}{k+2}(t-r-\alpha+k)$, then

$$\int_{\frac{t-r-\alpha}{2}}^{\frac{t+r-\alpha}{2}}\frac{\lambda^{1-p}}{(2\lambda+k)^l}\mathrm{d}\lambda\le\frac{(\frac{k+2}{2})^{p+l-2}}{2^{2-p}(l+p-2)}(t-r-\alpha+k)^{-p-l+2}.$$

From Cases 1 and 2, there exists a constant N_1 such that

$$\int_{t-r}^{t+r}\frac{(\frac{\beta-\alpha}{2})^{1-p}}{(\beta+2k)^l}\frac{1}{2}\mathrm{d}\beta\le N_1(t-r-\alpha+k)^{2-p-l}.$$

Now we can estimate first integral in (9) and get

$$\begin{aligned}&\int_{-k}^{\frac{t-r}{2}}(\alpha+2k)^{-p(2p-3)}\mathrm{d}\alpha\int_{t-r}^{t+r}\frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}\mathrm{d}\beta=\int_{-k}^{\frac{t-r}{2}}(\alpha+2k)^{-p(2p-3)}N_1(t-r-\alpha+k)^{2-p-p}\mathrm{d}\alpha\\&\le N_3\left(\frac{t-r}{2}+k\right)^{2-2p}\int_{-k}^{\frac{t-r}{2}}(\alpha+2k)^{-p(2p-3)}\mathrm{d}\alpha\le N_1\left(\frac{t-r+2k}{2}\right)^{2-2p}\int_{-k}^{\infty}(\alpha+2k)^{-p(2p-3)}\mathrm{d}\alpha\\&=\frac{N_1k^{1-p(2p-3)}}{(p(2p-3)-1)2^{2-2p}}(t-r+2k)^{2-2p}.\end{aligned}$$

The second integral in (9) can be estimated as follows

$$\begin{aligned}&\int_{\frac{t-r}{2}}^{t-r}(\alpha+2k)^{-p(2p-3)}\mathrm{d}\alpha\int_{t-r}^{t+r}\frac{\left(\frac{\beta-\alpha}{2}\right)^{1-p}}{(\beta+2k)^p}\frac{1}{2}\mathrm{d}\beta\\&\le(\frac{t-r}{2}+2k)^{-p(2p-3)}\int_{\frac{t-r}{2}}^{t-r}(t-r+2k)^{-p+l}\mathrm{d}\alpha\int_{t-r}^{t+r}\frac{(\frac{\beta-\alpha}{2})^{1-p}}{(\beta+2k)^l}\frac{1}{2}\mathrm{d}\beta\\&\le(\frac{t-r}{2}+2k)^{-p(2p-3)}(t-r+2k)^{-p+l}N_1\int_{\frac{t-r}{2}}^{t-r}(t-r-\alpha+k)^{2-l-p}\mathrm{d}\alpha\\&\le 2^{p(2p-3)}N_1(t-r+2k)^{-p(2p-3)-p+l}\int_{-\infty}^{t-r}(t-r-\alpha+k)^{2-l-p}\mathrm{d}\alpha\\&\le\frac{2^{p(2p-3)}N_1k^{3-l-p}}{l+p-3}(t-r+2k)^{-p(2p-3)-p+l},\end{aligned}$$

where $l=3-p+p\delta_1$, $-p(2p-3)-p+l<-2p+2$ and δ_1 is defined in Section 5.

From the estimates of first integral and second integral of (9), the result follows.

Lemma 6 *Let $u\in\Omega_{p,k}$, there exists a constant C_4 such that for $0<r<t+k$, $\alpha=1$ and 2,*

$$\int_0^t(r+t-\tau)^\alpha|G(u(r+t-\tau,\tau))|\mathrm{d}\tau\le C_4\frac{||u||^p}{(t-r+2k)^{2p-2}},$$

$$\int_0^t |r-t+\tau|^{\alpha}|G(u(r-t+\tau,\tau))|\mathrm{d}\tau \le C_4\frac{||u||^p}{(t-r+2k)^{2p-2}},$$
$$\int_0^t (r+t-\tau)^2|\frac{\partial G}{\partial r}(u(r+t-\tau,\tau))|\mathrm{d}\tau \le C_4\frac{||u||^p}{(t-r+2k)^{2p-2}},$$
$$\int_0^t (r-t+\tau)^2|\frac{\partial G}{\partial r}(u(r-t+\tau,\tau))|\mathrm{d}\tau \le C_4\frac{||u||^p}{(t-r+2k)^{2p-2}}.$$

Proof Note that $\lambda^{\alpha}|G(u(\lambda,\tau))| \le A\lambda^{\alpha}|u(\lambda,\tau)|^p \le A||u||^p\frac{\lambda^{\alpha}z^p(\lambda)}{\varphi^p(\lambda,\tau)}$, we have

$$\int_0^t (r+t-\tau)^{\alpha}|G(u(r+t-\tau,\tau))|\mathrm{d}\tau = \int_{t_2}^t \lambda^{\alpha}|G(u(\lambda,\tau))|\Big|_{\lambda=r+t-\tau}\mathrm{d}\tau$$
$$\le \int_{t_2}^t A||u||^p\frac{\lambda^{\alpha}z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}\mathrm{d}\tau \le A||u||^p\int_{t_2}^t\frac{\lambda^{\alpha}z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}\mathrm{d}\tau.$$

By Lemma 3 and Lemma 4, we get the first inequality.

Similarly, we can get the second inequality in the lemma.

Note that

$$\lambda^2\left|\frac{\partial G}{\partial u}(u(\lambda,\tau))\right| = \lambda^2\left|\frac{\partial G}{\partial r}\cdot\frac{\partial u}{\partial r}\right| \le \lambda^2A|u|^{p-1}|u_r| \le A\lambda^2\left(\frac{z(\lambda)||u||}{\varphi(\lambda,\tau)}\right)^{p-1}\frac{2||u||}{\lambda\varphi(\lambda,\tau)}$$
$$\le A||u||^p\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}.$$

By Lemma 4, the third and fourth inequalities follow from Lemma 4.

Lemma 7 *Let $u\in\Omega_{k,p}$ and*

$$T_1u(r.t) = \frac{C_0}{r^3}T_0u(r,t) = \frac{C_0}{r^3}\int_0^t\int_{|r-t+\tau|}^{r+t-\tau}\lambda(\lambda^2+r^2-(t-\tau)^2)G(u(\lambda,\tau))\mathrm{d}\lambda\mathrm{d}\tau.$$

Then T_1u, rT_1u and $(rT_1u)_r$ are continuous, and the following inequalities hold

$$|T_1u(r,t)| \le C_5\frac{||u||^p}{\varphi(r,t)},\quad |rT_1u(r,t)| \le C_5\frac{||u||^p}{\varphi(r,t)},\quad |(rT_1u)_r(r,t)| \le C_5\frac{||u||^p}{\varphi(r,t)}.$$

Proof Since $\lambda|G(u(\lambda,\tau))| \le A||u||^p\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}$ and $\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}$ is integrable in λ. Also $\frac{\partial}{\partial r}(\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau)) = 2r\lambda G(u(\lambda,\tau))$ is integrable in λ. Therefore

$$(T_0u)_r(r,t) = \int_0^t\int_{r-t+\tau}^{r+t-\tau}2r\lambda G(u(\lambda,\tau))\mathrm{d}\lambda\mathrm{d}\tau + \int_0^t\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau))\Big|_{\lambda=r-t+\tau}^{\lambda=r+t-\tau}\mathrm{d}\tau$$

is continuous.

Note that $\lambda(r^2+\lambda^2-(t-\tau)^2)G(u(\lambda,\tau))|_{\lambda=r-t+\tau}^{\lambda=r+t-\tau} = 2r\lambda^2G(u(\lambda,\tau))|_{\lambda=r=t+\tau}^{\lambda=r+t-\tau}\mathrm{d}\tau$, then

$$(T_0u)_r(r,t) = 2rJu(r,t),$$

where $Ju(r,t) = \int_0^t\int_{r-t+\tau}^{r+t-\tau}\lambda G(u(\lambda,\tau))\mathrm{d}\lambda\mathrm{d}\tau + \int_0^t\lambda^2G(u(\lambda,\tau))|_{\lambda=r-t+\tau}^{\lambda=r+t-\tau}\mathrm{d}\tau$.

Therefore, Ju is continuous, and by Lemmas 5,6 there exists a constant M such that

$$|Ju(r,t)| \leq M\frac{||u||^p}{(t-r+2k)^{2p-2}} \tag{10}$$

Differentiate Ju with respect to r, we have

$$(Ju)_r(r,t) = \int_0^t [3(r+t-\tau)G(u(r+t-\tau,\tau)) - 3(r-t+\tau)G(u(r-t+\tau,\tau))+$$
$$(r+t-\tau)^2\frac{\partial}{\partial r}(G(u(r+t-\tau,\tau)) - (r-t+\tau)^2\frac{\partial}{\partial r}(G(u(r-t+\tau,\tau))].$$

By Lemma 6, we can conclude that for $0<r<t+k$

$$|(Ju)_r(r,t)| \leq 8C_4\frac{||u||^p}{(t-r+2k)^{2p-2}}\ . \tag{11}$$

Since $Ju(0,t)=0$ and $T_0u(0,t)=0$, we have, using (10) and (11)

$$|Ju(r,t)| = |\int_0^r (Ju)_r(s,t)\mathrm{d}s| \leq \int_0^r |(Ju)_r(s,t)|\mathrm{d}s \leq 8C_4||u||^p\frac{r}{(t-r+2k)^{2p-2}} \tag{12}$$

$$T_0u(r,t) = \int_0^r (T_0u)_r(s,t)\mathrm{d}s = \int_0^r 2sJu(s,t)\mathrm{d}s. \tag{13}$$

Hence the following inequalities is true

$$T_1u(r,t) = \frac{C_0}{r^3}T_0u(r,t) = \frac{C_0}{r^3}\int_0^r 2sJu(s,t)\mathrm{d}s = \frac{2C_0}{r^3}\int_0^r 2s\int_0^s (Ju)_s(\lambda,t)\mathrm{d}\lambda\mathrm{d}s,$$

which implies that T_1u, rT_1u are continuous.

To estimate rT_1u, first we need to estimate the integral $\frac{2C_0}{r^2}\int_0^r |Ju(s,t)|\mathrm{d}s$.

If $t-r+2k > \frac{1}{2}(t+r+2k)$,

$$\frac{2C_0}{r^2}\int_0^r |Ju(s,t)|\mathrm{d}s \leq \frac{2C_0}{r^2}\int_0^r 8C_4||u||^p\frac{s}{(t-s+2k)^{2p-2}}\mathrm{d}s$$
$$\leq \frac{16C_0C_4||u||^p}{r^2}\frac{r^2}{(t-r+2k)^{2p-2}} \leq 32C_0C_4\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

If $t-r+2k \leq \frac{1}{2}(t+r+2k)$, i.e., $r \geq \frac{r+t+2k}{3}$, then

$$\frac{2C_0}{r^2}\int_0^r |Ju(s,t)|\mathrm{d}s \leq \frac{2C_0}{r^2}\int_0^r \frac{M||u||^p}{(t-r+2k)^{2p-2}}$$
$$\leq 2C_0M\frac{1}{r}\frac{||u||^p}{(t-r+2k)^{2p-2}} \leq \frac{8C_0M}{k}\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}. \tag{14}$$

Now we estimate rT_1u

$$|rT_1u(r,t)| = \frac{C_0}{r^2}|T_0u(r,t)| \leq \frac{C_0}{r^2}\int_0^r 2s|Ju(s,t)|\mathrm{d}s \leq \frac{2C_0}{r}\int_0^r |Ju(s,t)|\mathrm{d}s \tag{15}$$

If $t-r+2k>\frac{1}{2}(t+r+2k)$, then

$$\frac{2C_0}{r}\int_0^r |Ju(s,t)|\mathrm{d}s \le \frac{2C_0}{r}\int_0^r \frac{M||u||^p}{(t-s+2k)^{2p-2}}\mathrm{d}s$$
$$\le 2C_0M\frac{||u||^p}{(t-r+2k)^{2p-2}} \le 4C_0M\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

If $t-r+2k\le\frac{1}{2}(t+r+2k)$,

$$\frac{2C_0}{r}\int_0^r \frac{M||u||^p}{(t-s+2k)^{2p-2}}\mathrm{d}s = \frac{2C_0M}{(2p-3)r}\left(\frac{||u||^p}{(t-r+2k)^{2p-3}} - \frac{||u||^p}{(t+2k)^{2p-3}}\right)$$
$$\le \frac{2C_0M}{(2p-3)}\frac{||u||^p}{r(t-r+2k)^{2p-3}} \le \frac{8C_0M}{2p-3}\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

For $(rT_1u)_r$, we have $(rT_1u)_r(r,t) = -\frac{3C_0}{r^3}T_0u(r,t)+\frac{C_0}{r^2}(T_0u)_r(r,t)$. We have proved that the first term is continuous and satisfies the estimate. The second term is also continuous, since

$$\frac{C_0}{r^2}(T_0u)_r(r,t) = \frac{C_0}{r}Ju(r,t) = \frac{C_0}{r}\int_0^r (Ju)_r(s,t)\mathrm{d}s.$$

If $t-r+2k>\frac{1}{2}(t+r+2k)$, then by (12)

$$\frac{2C_o}{r}|Ju(r,t)| \le \frac{2C_0}{r}\frac{8C_4||u||^p r}{(t-r+2k)^{2p-2}} \le 32C_0C_4\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

If $t-r+2k\le\frac{1}{2}(t+r+2k)$, then by (10)

$$\frac{2C_0}{r}|Ju(r,t)| \le \frac{2C_0}{r}\frac{M||u||^p}{(t-r+2k)^{2p-2}} \le \frac{8C_0M}{k}\frac{||u||^p}{(t+r+2k)(t-r+2k)^{2p-3}}.$$

4. Proof of Theorem 1

We will prove T is a contract map and maps a proper chosen set to itself.

By Lemma 7 we have

$$||Tu|| \le ||u^0|| + ||T_1u|| \le ||u^0|| + 3C_5||u||^p.$$

Let $\varepsilon = (6C_5)^{-\frac{1}{p-1}}$, hence $C_5 = \frac{1}{6}\varepsilon^{1-p}$. Let $\Omega^\varepsilon_{p,k} = \{u\in\Omega_{p,k}||u||\le\varepsilon\}$.

Choose $\sum_{i=0}^2(||f^{(i)}||_\infty + ||g^{(i)}||_\infty)$ small enough such that $||u^0||<\frac{1}{2}\varepsilon$, then

$$||Tu|| \le ||u^0|| + 3C_5||u||^p \le \frac{\varepsilon}{2} + 3\cdot\frac{1}{6}\varepsilon^{1-p}\varepsilon^p = \varepsilon.$$

Therefore T maps $\Omega^\varepsilon_{p,k}$ to itself. Exactly same estimate as before but replace G by $A(|u|^p-|v|^p)$ and note that $|\,|u|^p-|v|^p| \le (|u|+|v|)^{p-1}|u-v|$, then

$$||Tu-Tv|| \le 3C_5(||u||+||v||)^{p-1}||u-v|| \le 3C_5(2\varepsilon)^{p-1}||u-v|| = 2^{p-2}||u-v||.$$

Since $p < 2$, hence $2^{p-2} < 1$, and T is a contraction map on $\Omega^{\epsilon}_{p,k}$, therefore T is a contraction map on $\Omega^{\epsilon}_{p,k}$. By contraction mapping theorem, there exists a unique fixed point u of $\Omega^{\epsilon}_{p,k}$ such that $Tu = u$, i.e., the global solution of the nonlinear wave equation exists.

Theorem 3 *For $n = 5$ and $p \geq 2$, the global solution of the nonlinear wave equation exists in $\Omega_{p,k}$, where $\Omega_{p,k}$ defined as before except $\varphi(r,t) = (t+r+2k)(t-r+2k)^{p-1}$.*

Proof The proof is similar to the proof of Theorem 1, and is omitted here.

5. Proof of Lemma 3 and Lemma 4

Proof of Lemma 3 Without loss of generality, we assume $t - r \geq 0$.

Case 1 If $r - t + \tau > 0$, i.e., $\tau > t - r$, then

$$\varphi(r-t+\tau,\tau) = (r-t+\tau+\tau+2k)(t-r+2k)^{2p-3} = (2\tau+r-t+2k)(t-r+2k)^{2p-3}$$

$$\begin{aligned}
&\int_{t_1}^{t} \frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau} \mathrm{d}\tau = \int_0^r \frac{\lambda z^p(\lambda)}{(2\lambda+t-r+2k)^p(t-r+2k)^{p(2p-3)}}\mathrm{d}\lambda \\
&\leq (t-r+2k)^{-p(2p-3)}\left[\int_0^1 \frac{\lambda}{(2\lambda+t-r+2k)^p}\mathrm{d}\lambda + \int_1^\infty \frac{\lambda^{1-p}}{(2\lambda+t-r+2k)^p}\mathrm{d}\lambda\right] \\
&\leq (t-r+2k)^{-p(2p-3)}\left[(t-r+2k)^{-p} + (t-r+2k)^{-\frac{p}{2}}\int_1^\infty \lambda^{1-\frac{3}{2}p}\mathrm{d}\lambda\right] \\
&\leq (t-r+2k)^{-p(2p-3)-\frac{p}{2}}\left[(t-r+2k)^{-\frac{p}{2}} + \frac{1}{\frac{3}{2}p-2}\right] \leq \left(k^{-\frac{p}{2}} + \frac{1}{\frac{3}{2}p-2}\right)(t-r+2k)^{-p(2p-3)-\frac{p}{2}}.
\end{aligned}$$

For $p > \frac{3+\sqrt{17}}{4}$, we have $-p(2p-3) - \frac{p}{2} < -2p+2$ and note that $t - r + 2k > k$ then

$$\begin{aligned}
(t-r+2k)^{-p(2p-3)-\frac{p}{2}} &= (t-r+2k)^{-2p+2}(t-r+2k)^{-p(2p-3)-\frac{p}{2}+2p-2} \\
&\leq k^{-p(2p-3)-\frac{p}{2}+2p-2}(t-r+2k)^{-2p+2}.
\end{aligned}$$

Case 2 If $r-t+\tau < 0$, i.e., $\tau < t-r$, then $\varphi(t-r-\tau,\tau) = (t-r+2k)(2\tau-(t-r)+2k)^{2p-3}$

$$\int_{t_1}^{t-r} \frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t-r-\tau}\mathrm{d}\tau = \int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda$$

If $t - r \leq k$, then $t_1 = \max\{0, \frac{1}{2}(t-r-k)\} = 0$

$$\begin{aligned}
&\int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda \\
&\leq \int_0^{t-r} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2(t-r)+2k)^{p(2p-30}}\mathrm{d}\lambda \\
&\leq (t-r+2k)^{-p}\int_0^k \frac{\lambda z^p(\lambda)}{(-(t-r)+2k)^{p(2p-3)}} \\
&\leq (t-r+2k)^{-p}\int_0^k \frac{\lambda}{k^{p(2p-3)}}\mathrm{d}\lambda \leq (t-r+2k)^{-p}k^{2-p(2p-3)}.
\end{aligned}$$

Since $p < 2$, we have $(t-r+2k)^{2-p} \geq k^{2-p}$. Therefore

$$\int_{t_1}^{t-r} \frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t-r-\tau} \mathrm{d}\tau \leq (t-r+2k)^{-2p+2} k^{-p(2p-3)+p}.$$

If $t-r > k$, then $t_1 = \frac{1}{2}(t-r-k) > 0$

$$\int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda)^{p(2p-3)}}\mathrm{d}\lambda \leq \int_0^{t-r-t_1} \frac{\lambda^{1-p}}{(t-r+2k)^p(t-r-2\lambda)^{p(2p-3)}}\mathrm{d}\lambda$$

$$\leq (t-r+2k)^{-p}\left[\int_0^{\frac{k}{2}} + \int_{\frac{k}{2}}^{\frac{t-r+2k}{4}} + \int_{\frac{t-r+2k}{4}}^{\frac{t-r+k}{2}}\right] \frac{\lambda^{1-p}}{(t-r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda$$

$$\leq (t-r+2k)^{-p}(t-r+k)^{-p(2p-3)} \int_0^{\frac{k}{2}} \lambda^{1-p}\mathrm{d}\lambda +$$

$$(t-r-2(\frac{t-r+2k}{4})+2k)^{-p(2p-3)} \int_{\frac{k}{2}}^{t-r+2k} \lambda^{1-p}\mathrm{d}\lambda +$$

$$(\frac{t-r+2k}{4})^{-p+1} \int_{\frac{t-r+2k}{2}}^{\frac{t-r+k}{2}} (t-r-2\lambda+2k)^{-p(2p-3)}\mathrm{d}\lambda$$

Since

$$\int_0^{\frac{k}{2}} \lambda^{1-p}\mathrm{d}\lambda = \frac{1}{p-1}(\frac{k}{2})^{2-p},$$

$$\int_{\frac{k}{2}}^{\frac{t-r+2k}{4}} \lambda^{1-p}\mathrm{d}\lambda = (\frac{k}{2})^{1-p}\left(\frac{t-r+2k}{4} - \frac{k}{2}\right) \leq (\frac{k}{2})^{1-p}\frac{t-r+2k}{4},$$

$$\int_{\frac{t-r+2k}{4}}^{\frac{t-r+k}{2}} (t-r-2\lambda+2k)^{-p(2p-3)} \leq \int_{-\infty}^{\frac{t-r+k}{2}} (t-r-2\lambda+2k)^{-p(2p-3)}\mathrm{d}\lambda$$

$$= \frac{1}{2}\int_k^{\infty} \alpha^{-p(2p-3)}\mathrm{d}\alpha = \frac{1}{2}\frac{k^{1-p(2p-3)}}{p(2p-3)-1}.$$

Note that $t-r+k = \frac{t-r}{2} + \frac{t-r}{2} + k > \frac{t-r}{2} + \frac{3k}{2} > \frac{t-r+2k}{2}$, $t-r+2k > 3k$, $-p(2p-3) < -p+2$, and $-p(2p-3)+1 < -p+2$, therefore there exists a constant, say M_0, such that for $t-r \leq k$

$$\int_0^{t-r-t_1} \frac{\lambda z^p(\lambda)}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda \leq M_0(t-r+2k)^{-2p+2}.$$

Combine all above, there exists a constant M_1 such that

$$\int_{t_1}^{t} \frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=|t-r+\tau|} \mathrm{d}\tau \leq \frac{M_1}{(t-r+2k)^{2p-2}}.$$

Now we prove the second inequality. First we consider $t+r>k$, then $t_2=\frac{1}{2}(t+r-k)>0$ and $\varphi(t+r-\tau,\tau)=(t+r+2k)(2\tau-(t+r)+2k)^{2p-3}$,

$$\begin{aligned}\int_{t_2}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau}\mathrm{d}\tau &= \int_{r}^{r+t-t_2}\frac{\lambda z^p(\lambda)}{(t+r+2k)^p(t+r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda\\ &\le \int_{r}^{r+t-t_2}\frac{\lambda^{1-p}}{(t+r+2k)^p(t+r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda\\ &\le (t+r+2k)^{-p}\left[\int_0^{\frac{k}{2}}+\int_{\frac{k}{2}}^{\frac{t+r+2k}{4}}+\int_{\frac{t+r+2k}{4}}^{\frac{t+r+k}{2}}\right]\frac{\lambda^{1-p}}{(t+r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda.\end{aligned}$$

A similar argument used in the proof of first inequality, there exists a constant M_0' such that for $t+r>k$,

$$\int_{t_2}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau}\mathrm{d}\tau\le\frac{M_0'}{(t+r+2k)^{2p-2}}\le\frac{M_0'}{(t-r+2k)^{2p-2}}.$$

If $t+r\le k$, then $t_2=0$ and

$$\begin{aligned}&\int_{t_2}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau}\mathrm{d}\tau\le\int_{0}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau}\mathrm{d}\tau\le\int_{0}^{t}\frac{\lambda^{1-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau}\mathrm{d}\tau\\ &=\int_0^t\frac{(t+r-\tau)^{1-p}}{(t+r+2k)^p(2\tau-r-t+2k)^{p(2p-3)}}\mathrm{d}\tau\le(t+r+2k)^{-p}k^{-p(2p-3)}\int_0^t(r+t-\tau)^{1-p}\mathrm{d}\tau\\ &=(t-r+2k)^{-p}k^{-p(2p-3)}\left[-\frac{1}{2-p}r^{2-p}+\frac{1}{2-p}(r+t)^{2-p}\right]\\ &\le(t+r+2k)^{-p}k^{-p(2p-3)}\frac{1}{2-p}(r+t)^{2-p}\le\frac{k^{-p(2p-3)}}{2-p}(t+r+2k)^{-2p+2}.\end{aligned}$$

Therefore there exists a constant M_1' such that for $0<r<t+k$

$$\int_{t_2}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}\mathrm{d}\tau\le\frac{M_1'}{(t+r+2k)^{2p-2}}\le\frac{M_1'}{(t-r+2k)^{2p-2}}.$$

Proof of Lemma 4 Note $p\in(\frac{3+\sqrt{17}}{4},2)$, then there exists a number $\delta_0>0$ such that

$$p=\frac{3+\delta_0+\sqrt{(3+\delta_0)^2+8}}{4}.$$

For $r-t+\tau>0$, i.e., $\tau>t-r$,

$$\begin{aligned}\int_{t-r}^{t}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau}\mathrm{d}\tau=&(t-r+2k)^{-p(2p-3)}\int_0^r\frac{\lambda^{2-p}}{(2\lambda+t-r+2k)^p}\mathrm{d}\lambda\\ \le&(t-r+2k)^{-p(2p-3)}[(t-r+2k)^{-p}\int_0^1\lambda^{2-p}\mathrm{d}\lambda+\\ &\int_1^{\infty}\frac{\lambda^{2-p}}{(2\lambda+t-r+2k)^{(3-p-\delta_1p)+p-(3-p-\delta_1p)}}\mathrm{d}\lambda],\end{aligned}$$

where $0<\delta_1<\min\{\delta_0, 2-\frac{3}{p}\}$, it is easy to see $p-(3-p+\delta_1 p)>0$ and

$$\int_1^\infty \frac{\lambda^{2-p}}{(2\lambda+t-r+2k)^{(3-p+\delta_1 p)+p-(3-p+\delta_1 p)}}\mathrm{d}\lambda \le (t-r+2k)^{-2p+3+\delta_1 p}\int_1^\infty \lambda^{-1-\delta_1 p}\mathrm{d}\lambda$$
$$= \frac{1}{p\delta_1}(t-r+2k)^{-2p+3+\delta_1 p}.$$

Therefore

$$\int_{t-r}^t \frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau}\mathrm{d}\tau \le (t-r+2k)^{-p(2p-3)}\left[(t-r+2k)^{-p}+\frac{1}{p\delta_1}(t-r+2k)^{-2p+3+p\delta_1}\right]$$
$$=(t-r+2k)^{-p(2p-3)-2p+3+p\delta_1}\left[(t-r+2k)^{p-3-p\delta_1}+\frac{1}{p\delta_1}\right]$$
$$\le \left[k^{p-3-p\delta_1}+\frac{1}{p\delta_1}\right](t-r+2k)^{-p(2p-3)-2p+3+p\delta_1}.$$

Since $p=\frac{3+\delta_0+\sqrt{(3+\delta_0)^2+8}}{4}$, we have $-p(2p-3)-2p+3+p\delta_1<-2p+2$, i.e., $2p^2-(3+\delta_1)p-1>0$, which requires $p>\frac{3+\delta_1+\sqrt{(3+\delta_1)^2+8}}{4}$. Therefore

$$\int_{t-r}^t \frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau}\mathrm{d}\tau \le \left[k^{3-p-p\delta_1}+\frac{1}{p\delta_1}\right](t-r+2k)^{-2p+2}.$$

For $r-t+\tau<0$, i.e., $\tau<t-r$,

$$\int_{t_1}^{t-r} \frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t-r-\tau}\mathrm{d}\tau = \int_0^{t-r-t_1}\frac{\lambda^{2-p}}{(t-r+2k)^p(t-r-2\lambda+2k)^{p-(2p-3)}}\mathrm{d}\lambda.$$

If $t-r\le k$, then $t_1=0$ and

$$\int_0^{t-r}\frac{\lambda^{2-p}}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}}\mathrm{d}\lambda$$
$$\le \int_0^{t-r}\frac{\lambda^{2-p}}{(t-r+2k)^p(t-r-2(t-r)+2k)^{p(2p-3)}}\mathrm{d}\lambda$$
$$\le (t-r+2k)^{-p}\int_0^k\frac{\lambda^{2-p}}{k^{p(2p-3)}}\mathrm{d}\lambda \le (t-r+2k)^{-p}k^{3-p-p(2p-3)}.$$

Note that $(t-r+2k)^{2-p}\ge k^{2-p}$, we have

$$\int_{t_1}^{t-r} \frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau}\mathrm{d}\tau \le k^{1-p(2p-3)}(t-r+2k)^{-2p+2}$$

for all $t-r\le k$. If $t-r>k$, then $t_1=\frac{1}{2}(t-r-k)>0$,

$$\int_{t_1}^{t-r}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r-t+\tau}d\tau=\int_0^{t-r-t_1}\frac{\lambda^{2-p}}{(t-r+2k)^p(t-r-2\lambda+2k)^{p(2p-3)}}d\lambda$$

$$\le(t-r+2k)^{-p}[(t-r+k)^{-p(2p-3)}\int_0^{\frac{k}{2}}\lambda^{2-p}d\lambda+$$

$$(t-r-2(\frac{t-r+2k}{4})+2k)^{-p(2p-3)}\int_{\frac{k}{2}}^{\frac{t-r+2k}{4}}\lambda^{2-p}d\lambda+$$

$$(\frac{t-r+k}{2})^{-p+2}\int_{\frac{t-r+2k}{4}}^{\frac{t-r+k}{2}}(t-r-2\lambda+2k)^{-p(2p-3)}d\lambda].$$

Note that $\int_{\frac{k}{2}}^{\frac{t-r+2k}{4}}\lambda^{2-p}d\lambda\le\frac{1}{3-p}(\frac{t-r+2k}{4})^{3-p}$ and $-p(2p-3)-p+3\le-p+2$, therefor there exists a constant N_0 such that

$$\int_{t_1}^{t-r}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t-r-\tau}d\lambda\le N_0(t-r+2k)^{-2p+2}$$

for $t-r>k$. Combining all estimates gives

$$\int_{t_1}^{t}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=|r-t+\tau|}d\lambda\le C_2(t-r+2k)^{-2p+2}\ .$$

Now let us turn our attention to the second inequality in the lemma If $t+r\le k$, the proof is similar to that of $\int_{t_2}^{t}\frac{\lambda z^p(\lambda)}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}d\tau\le C_1(t+r+2k)^{-2p+2}$ and is omitted.

If $t+r>k$, then $t_2=\frac{1}{2}(t+r+k)>0$ and $\varphi(t+r-\tau,\tau)=(t+r+2k)(2\tau-(t+r)+2k)^{2p-3}$,

$$\int_{t_2}^{t}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=t+r-\tau}d\tau=\int_r^{r+t-t_2}\frac{\lambda^{2-p}}{(t+r+2k)^p(t+r-2\lambda+2k)^{p(2p-3)}}d\lambda,$$

$$\le(t+r+2k)^{-p}\left[\int_0^{\frac{k}{2}}+\int_{\frac{k}{2}}^{\frac{t+r+2k}{4}}+\int_{\frac{t+r+2k}{4}}^{\frac{t+r+k}{2}}\right]\frac{\lambda^{2-p}}{(t+r-2\lambda+2k)^{p(2p-3)}}d\lambda$$

$$\le(t+r+2k)^{-p}[(t+r+k)^{-p(2p-3)}\int_0^{\frac{k}{2}}\lambda^{2-p}d\lambda+$$

$$(\frac{t+r+2k}{2})^{-p(2p-3)}\int_{\frac{k}{2}}^{\frac{t+r+2k}{4}}\lambda^{2-p}d\lambda+(\frac{t+r+k}{2})^{2-p}\int_{\frac{t+r+2k}{4}}^{\frac{t+r+k}{2}}(t+r-2\lambda+2k)^{-p(2p-3)}d\lambda]$$

$$\le(t+r+2k)^{-p}[(t+r+k)^{-p(2p-3)}\left(\frac{k}{2}\right)^{-p(2p-3)+3-p}+\left(\frac{t+r+2k}{2}\right)^{3-p}+$$

$$\frac{k^{1-p(2p-3)}}{p(2p-3)-1}\left(\frac{t+r+k}{2}\right)^{2-p}]\ .$$

Note that $-p-p(2p-3)<-2p+2$ and $-p-(2p-3)+3-p<-2p+2$ (which requires $p>\frac{3+\sqrt{17}}{4}$), then we conclude there exists a constant N_1' such that for $t+r>k$

$$\int_{t_2}^{t}\frac{\lambda^{2-p}}{\varphi^p(\lambda,\tau)}\Big|_{\lambda=r+t-\tau}d\tau<N_1'(t+r+2k)^{-2p+2}<N_1'(t-r+2k)^{-2p+2}\ .$$

This concludes the proof of the lemma.

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五维空间中半线性波动方程整体解的存在性定理

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摘 要: 本文研究五维空间中半线性波动方程 $u_{tt} - \Delta u = G(u)$ 整体解的存在性，其中 $G(u) \sim |u|^p$ 并且 $p > \frac{3+\sqrt{17}}{4}$. 利用经典的迭代方法证明了：如果初始值很小并且紧支的，径向对称方程有一个经典整体解.