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Derivations on Semiprime Rings *

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Abstract: Let R be a semiprime ring with the center $Z(R)$, d and g be derivations of R, L be a nonzero left ideal of R and $r_R(L)=0$. Suppose that $d(x)x-xg(x)\in Z(R)$ for all $x\in L$, then $d(R)\subseteq Z(R)$ and the ideal of R generated by $d(R)$ is in the center of R.

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1. Introduction and preliminaries

In [2] E. C. Posner initiated the study of centralizing and commuting mapping. He showed that if d is a centralizing derivation of a prime ring R, then either $d=0$ or R is commutative. Over the last twenty years a lot of work has been done on centralizing and commuting mappings. In [3] M. Brešar discussed a generalized case about derivation, he proved that if derivations d and g of prime ring satisfy $d(x)x-xg(x)\in Z(R)$ for all $x\in L$, L be a nonzero left ideal of R and d is nonzero, then R is commutative. When R is a semiprime ring and $d=g$, T. K. Lee[4] and C. Lanski[5] have given some interesting results.

The main goal of this paper is to prove that if a pair of derivation d and g in a semiprime ring satisfy $d(x)x-xg(x)\in Z(R)$ for all $x\in L$, L is a nonzero left ideal of R and $r_R(L)=0$, then $d(R)\subseteq Z(R)$ and the ideal of R generated by d(R) is in the center of R.

Throughout this paper, R will be a semiprime ring with the center $Z(R)$, $Q_{mr}(R)$ be the maximal right quotient ring of R, C be the extended centroid of R, $B(C)$ be the subset of all idempotents of C. Spec(B) denotes the set of all maximal ideal of the Boolean ring $B(C)$. $O(R)$ is the orthogonal completion of R. If L is a left ideal of R, then $O(L)$ is a left ideal of $O(R)$. Let $M\in \mathrm{Spec}(B)$, ϕ_M: $Q_{mr}(R)\rightarrow Q_{mr}/Q_{mr}M=Q_M$ be the canonical surjective of rings (see [1, Chapter 3] for details).

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2. Results

Theorem *Let R be a semiprime ring, d and g be the derivations of R, L be a nonzero left ideal and $r_R(L) = 0$. Suppose that $d(x)x - xg(x) \in Z(R)$ for all $x \in L$, then $d(R) \subseteq Z(R)$ and the ideal of R generated by $d(R)$ is in the center of R.*

For the proof of the Theorem, we need the following Lemmas.

Lemma 1 *Let R be a semiprime ring, L be a nonzero left ideal of R and $r_R(L) = 0$, then $\varphi_M(O(L)) \neq 0$ for all $M \in \mathrm{Spec}(B)$.*

Proof First we show that $r_R(L) = 0$ implies $r_Q(L) = 0$. Let $q \in Q_{mr}$ such that $Lq = 0$, for this elenemt q, by [1, Th2.1.7(2)] there exists dense right ideal J of R such that $qJ \subseteq$R, clearly $LqJ = 0$, that is $qJ \in r_R(L)$, since $r_R(L)$=0, so we obtain $qJ = 0$, also [1, Th2.1.7(3)] implies $q = 0$, therefore $r_Q(L)$ is zero.

Next we show that $r_Q(L) = 0$ implies $r_Q(O(L)) = 0$, therefore $r_C(O(L)) = 0$. Since

$$O(L) = \{\sum_{u \in U}^{\perp} r_u u | U \text{ is dense orthogonal subset of } B \text{ and } \{r_u \mid u \in U\} \subseteq L\},$$

if there exists $q \in Q_{mr}$ such that O(L)q=0, take $rE(q) \in O(L)$ for all $r \in L$, then $0 = rE(q)q = rq$ for all $r \in L$, then q$\in r_Q(L)$, by $r_Q(L) = 0$, we get $q = 0$, so $r_Q(O(L)) = 0$, therefore $r_C(O(L)) = 0$.

Since $r_C(O(L)) = (1 - E(O(L))C = 0$, then $E(O(L)) = 1$. For all $M \in \mathrm{Spec}(B), E(O(L))$: $1 \notin M$, by [1, Corollary 3.2.4] we obtain $\phi_M(O(L)) \neq 0$ for all $M \in \mathrm{Spec}(B)$.

Lemma 2 *Let R be a semiprime ring, d and g be derivations of R, L be a nonzero left ideal of R and $r_R(L) = 0$. Suppose that $d(x)x - xg(x) \in Z(R)$ for all $x \in L$, then there exists $e \in B(C)$ such that eR is commutative and d induces a zero derivation on $(1 - e)R$.*

Proof Set $O(R)$ is the orthogonal completion of R

$$\Omega = \{+, -, \cdot, 0, \mathrm{Der}(R)\},$$

$\Delta_P = \{\| t_1 = t_2 \|, \| t \in T \| \; T$ is an orthogonal complete subset of $O(R)$ containing zero$\}$

Since $d(eq) = ed(q)$ for all $q \in Q_{mr}, d \in Der(R)$ then $O(R)$ is orthogonally complete $\Omega - \Delta - ring$, $O(R)_M = \phi_M(O(R))$ is a prime ring for all $M \in Spec(B)$, by Lemma 1 $\phi_M(O(L))$ is a nonzero left ideal of $O(R)_M$ for all $M \in \mathrm{Spec}(B)$. Consider formulas:

$$\begin{aligned} \Phi &= (\forall x)(\forall y) \, \| x \in O(L) \| \wedge \| [d(x)x - xg(x), y] = 0 \|, \\ \Psi_1 &= (\forall x)(\forall y) \, \| xy = yx \|, \\ \Psi_2 &= (\forall x) \, \| d(x) = 0 \| \, . \end{aligned}$$

It is easy to prove that Φ is a hereditery first order formula and Φ, Ψ_1, Ψ_2 are Horn formulas.

Since $[d(x)x - xg(x), y] = 0$ for all $x \in L$, $y \in R$ and

$$O(L) = \{\sum_{u \in U}^{\perp} r_u u | U \text{ is dense orthogonal subset of } B \text{ and } \{r_u \mid u \in U\} \subseteq L\},$$

by [1, Th3.1.8, Th3.1.9, Th3.1.14] we obtain

$$[d(x)x - xg(x), y] = 0 \text{ for all } x \in O(L), y \in O(R),$$

that is $O(R) \models \Phi$. By [1,Corollary 3.2.11], we have

$$O(R)_M \models \Phi = (\forall x)(\forall y) \parallel x \in \varphi_M(O(L)) \parallel \wedge \parallel [d(x)x - xg(x), y] = 0 \parallel .$$

Since for all $M \in Spec(B)$, $O(R)_M$ is a prime ring and $\varphi_M(O(L))$ is a nonzro left ideal of $O(R)_M$ by Lemma 1 and $Q(R)_M \models \Phi$ implies that $[d(x)x - xg(x), y] = 0$ for all $x \in \varphi_M(O(L))$, $y \in O(R)_M$. From [3, Th4.1], for every $M \in \mathrm{Spec}(B)$, either $O(R)_M$ is commutative or d is zero on $O(R)_M$, that is, $O(R)_M \models \Psi_1$ or $O(R)_M \models \Psi_2$. By[1, Th3.2.18] there exists $e \in B(C)$ such that $eO(R) \models \Psi_1$ and $(1-e)O(R) \models \Psi_2$, that is, $eO(R)$ is commutative and d induces a zero derivation on $(1-e)O(R)$. Since $R \subseteq O(R)$, then we complete the proof.

Proof of Theorem By Lemma 2 there exists $e \in B(C)$ such that eR is commutative and d induces a zero derivation on $(1-e)R$, then $[ex, ey] = 0$ and $d(r) = ed(r)$ for all $x, y, r \in R$, then

$$[x, y]d(r) = [x, y]ed(r) = [ex, ey]d(r) = 0 \text{ for all } x, y, r \in R,$$

$$d(r)[x, y] = ed(r)[x, y] = d(r)[ex, ey] = 0 \text{ for all } x, y, r \in R.$$

So $[R, R]d(R) = 0$ and $d(R)[R, R] = 0$, then by [5, Lemma 2] $d(R) \subseteq \mathrm{Ann}_R([R, R])$ the maximal central ideal of semiprime ring R, therefore $d(R) \subseteq Z(R)$ and the ideal of R generated by $d(R)$ is in the center of R.

From [3] we can obtain the following corollaries.

Corollary 1 *Let R be a semiprime ring and d derivation of R, L a nonzero left ideal of R and $r_R(L) = 0$. If d is centralizing or skew-centralizing on L, then $d(R) \subseteq Z(R)$ and the ideal of R generated by $d(R)$ is in the center of R.*

In particular, when $d(x) = [a, x]$ and d is skew-centralizing on L, i.e., $[a, x]x + x[a, x] \in Z(R)$ for all $x \in L$, from Corollary 1, we get $d(R) \in Z(R)$,then $d(x) = [a, x] \in Z(R)$ for all $x \in R$,by [6, Lemma 2],$a \in Z(R)$. This result can also be proved by using [5, Th 2], $[a, x]x + x[a, x] \in Z(R)$ for all $x \in L$ implies $[a, x^2] \in Z(R)$ for all x$\in L$, then $[a, x^2]_2 = 0$ for all $x \in L$, by [5, Th2] we obtain $[a, L] = 0$.Since $r_R(L) = 0$ and R contain no nonzero nilpotent left ideal, from $L(ar - ra)L = 0$ for all $r \in R$, we get $L(ar - ra) = 0$, therefore $ar = ra$ for all $r \in R$, so $a \in Z(R)$.

Corollary 2 *let R be a semiprime ring, L be a nonzero left ideal of R and a derivation d of R such that $[d(x) + ax + xb, x] \in Z(R)$ for all $x \in L$. then $(d - I_a)(R) \subseteq Z(R)$ and the ideal of R generated by $(d - I_a)(R)$ is in the center. If R is not commutative, then there exists a nonzero ideal K of R such that $d(x) = [x, a]$ for all $x \in K$.*

Proof Set $I_y = [, y]$ is inner derivation, then the relation $[d(x) + ax + xb, x] \in Z(R)$ implies $(d - I_a)(x)x - x(d + I_b)(x) \in Z(R)$, $d - I_a$ and $d + I_b$ are derivations of R, from main theorem, $(d-I_a)(R) \subseteq Z(R)$, and the ideal of R generated by $(d-I_a)(R)$ is in the center of R.

If R is not commutative, from Lemma 2, $e \neq 1$ then $1-e \neq 0$, by [1, Prop 2.2.3] there exists a dense ideal I of R such that $(1-e)I \subseteq$R. K=(1-e)I is a ideal of R and $K \subseteq (1-e)R$, then $(d-I_a)(K)=0$, therefor $d(x)=[x,a]$ for all $x \in K$.

Take d=0 in Corollary 2, we have

Corollary 3 *Let R be a semiprime ring, L be a nonzero left ideal of R and $r_R(L)=0$, if $a,b \in R$ are such that $x \mapsto ax+xb$ is centralizing on L, then $a \in Z(R)$.*

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半素环上的微商

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摘 要: 本文讨论了微商作用在半素环的某些左理想上的问题. 给出了如下结果：设 R 是带有中心 $Z(R)$ 的半素环. d 和 g 是 R 的微商, L 为 R 的非零左理想且 $r_R(L)=0$. 假设 $d(x)x-xg(x) \in Z(R)$ 对任意的 $x \in L$ 成立. 那么 $d(R) \subseteq Z(R)$ 且由 $d(R)$ 生成的 R 的理想在 R 的中心里.