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Some Equivalent Conditions for Block Independence of Reflexive Inner Inverse of Block Matrix *

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Abstract: The definition of block independence in generalized inverse of block matrix was introduced in [1]. In this paper, we give some equivalent conditions for two $m \times n$ matrices being block independent in reflexive inner inverse .

Key words: Reflexive inner inverse; rank; block independence.

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1. Introduction

Let $A \in C^{m\times n}$ and consider the following four Moore-Penrose equations:

$$AGA = A, \tag{1}$$

$$GAG = G, \tag{2}$$

$$(AG)^* = AG, \tag{3}$$

$$(GA)^* = GA. \tag{4}$$

Suppose $\mathcal{J} = \{i, j, \cdots, k\}$ is a nonempty subset of $\{1, 2, 3, 4\}$, then a matrix G is said to be a $\mathcal{J}$-inverse of A if G satisfies equation (i) for each $i \in \mathcal{J}$. The set of all $\mathcal{J}$-inverse of A is denoted by $A^{\{\mathcal{J}\}}$ and its any element is denoted by $A^{\mathcal{J}}$. $\{1\}$-inverse, $\{1,2\}$-inverse and $\{1,2,3,4\}$-inverse are also called *inner inverse*, *reflexive inner inverse* and M-P (Moore-Penrose) *inverse* of A, and are denoted by A^-, A^G and A^+, respectively.

Throughout this paper, all our matrices will be over the complex number field C. For matrix $A \in C^{m\times n}$, the symbols $rk(A), \mathcal{R}(A), \mathcal{RS}(A)$ denote the rank, the range (column space), the row space of A, respectively.

In the following, we suppose $\mathcal{J} = \{i, j, ..., k\}$ is a nonempty subset of $\{1, 2, 3, 4\}$. The following definition was given by Wang in [1].

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Definition 1.1 *Two $m \times n$ matrices B and C are block independent in $\mathcal{J}$-inverse if there exist $B^{\mathcal{J}} \in B\{\mathcal{J}\}, C^{\mathcal{J}} \in C\{\mathcal{J}\}$ such that*

$$\begin{pmatrix} B^{\mathcal{J}} & C^{\mathcal{J}} \end{pmatrix} \in \begin{pmatrix} B \\ C \end{pmatrix}^{\{\mathcal{J}\}} \quad \text{and} \quad \begin{pmatrix} B^{\mathcal{J}} \\ C^{\mathcal{J}} \end{pmatrix} \in \begin{pmatrix} B & C \end{pmatrix}^{\{\mathcal{J}\}}.$$

It should be noted that the definition of block independence of generalized inverse of block matrix in another meaning was given by Frank J.Hall etc in [2, 3], and the difference between these two kinds of definitions was pointed out in [1]. The necessary and sufficient conditions for two $m \times n$ matrices being block independent in reflexive inner inverse was also given in [1]. In the next section, we would give some equivalent conditions for two $m \times n$ matrices being block independent in reflexive inner inverse.

2. Main results

Theorem 2.1[4] *For $T \in C^{m\times n}$, denote $E_T = I - TT^-, F_T = I - T^-T$, then the following equations hold.*

$$rk(B \ \ C) = rk(B) + rk(E_B C) = rk(C) + rk(E_C B),$$

$$rk\begin{pmatrix} B \\ C \end{pmatrix} = rk(B) + rk(CF_B) = rk(C) + rk(BF_C).$$

Theorem 2.2 *Let $B, C \in C^{m\times n}$, then the followings are equivalent.*

(1) B, C are block independent in reflexive inner inverse;

(2) $rk(B + C) = rk(B) + rk(C)$;

(3) There exist nonsingular matrices P and Q such that

$$B = P\begin{pmatrix} I_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}Q, \qquad C = P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_C \end{pmatrix}Q,$$

where I_B and I_C are identity matrices of whatever size is appreciate to the rank of B and C, respectively;

(4) $\mathcal{R}(B + C) = \mathcal{R}(B) + \mathcal{R}(C)$;

(5) $\mathcal{RS}(B + C) = \mathcal{RS}(B) + \mathcal{RS}(C)$;

(6) There exist $B^G, C^G \in C^{n\times m}$ such that

$$B^G C = 0 \quad BC^G = 0 \quad C^G B = 0 \quad CB^G = 0;$$

(7) $rk(B \ \ C) = rk(B) + rk(C)$, $rk\begin{pmatrix} B \\ C \end{pmatrix} = rk(B) + rk(C)$;

(8) $\mathcal{R}(B) \cap \mathcal{R}(C) = \{0\}$, $\mathcal{RS}(B) \cap \mathcal{RS}(C) = \{0\}$;

(9) $rk(E_B C) = rk(C)$, $rk(CF_B) = rk(C)$;

(10) $rk(E_C B) = rk(B)$, $rk(BF_C) = rk(B)$.

Proof The equivalence between (2),(3),(4),(5),(6) can been seen in Theorem 2.1 in [5].

(6) $\Rightarrow$ (1) : If there exist $B^G, C^G \in C^{n\times m}$ such that

$$B^G C = 0, \quad BC^G = 0, \quad C^G B = 0, \quad CB^G = 0,$$

it can easily be verified that:

$$\begin{pmatrix} B^G & C^G \end{pmatrix} \in \begin{pmatrix} B \\ C \end{pmatrix}^{\{1,2\}} \quad \text{and} \quad \begin{pmatrix} B^G \\ C^G \end{pmatrix} \in \begin{pmatrix} B & C \end{pmatrix}^{\{12\}}.$$

So B, C are block independent in reflexive inner inverse.

(1) $\Rightarrow$ (6) : By

$$\begin{pmatrix} B^G & C^G \end{pmatrix} \in \begin{pmatrix} B \\ C \end{pmatrix}^{\{1,2\}},$$

one has

$$\begin{pmatrix} B^G & C^G \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix} \begin{pmatrix} B^G & C^G \end{pmatrix} = \begin{pmatrix} B^G & C^G \end{pmatrix}.$$

A short computation leads to

$$B^G BC^G = 0 \quad \text{and} \quad C^G CB^G = 0.$$

Left-multiplying the first equality by B, and the second equality by C, we obtain $BC^G = 0$ and $CB^G = 0$. Similarly, by

$$\begin{pmatrix} B^G \\ C^G \end{pmatrix} \in \begin{pmatrix} B & C \end{pmatrix}^{\{1,2\}},$$

one has $B^G C = 0$ and $C^G B = 0$.

(7) $\Leftrightarrow$ (8) : By $\mathcal{R}(B \quad C) = \mathcal{R}(B) + \mathcal{R}(C)$, we know that $rk(B \quad C) = rk(B) + rk(C)$ if and only if $\mathcal{R}(B) \cap \mathcal{R}(C) = \{0\}$. By

$$\mathcal{RS}\begin{pmatrix} B \\ C \end{pmatrix} = \mathcal{RS}(B) + \mathcal{RS}(C),$$

we have

$$rk\begin{pmatrix} B \\ C \end{pmatrix} = rk(B) + rk(C) \text{ if and only if } \mathcal{RS}(B) \cap \mathcal{RS}(C) = \{0\}.$$

(3) $\Rightarrow$ (7): If there exist nonsingular matrices P, Q, such that

$$B = P\begin{pmatrix} I_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q, \qquad C = P\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_C \end{pmatrix} Q,$$

then

$$rk(B\ \ C)=rk[P(B\ \ C)\begin{pmatrix}Q&0\\0&Q\end{pmatrix}]=rk\begin{pmatrix}I_B&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&I_C\end{pmatrix}=rk(B)+rk(C)$$

and

$$rk\begin{pmatrix}B\\C\end{pmatrix}=rk[\begin{pmatrix}P&0\\0&P\end{pmatrix}\begin{pmatrix}B\\C\end{pmatrix}Q]=rk\begin{pmatrix}I_B&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&I_C\end{pmatrix}=rk(B)+rk(C).$$

(7) $\Longrightarrow$ (2): First, there exist nonsingular matrices P_1, Q_1 such that

$$P_1BQ_1=\begin{pmatrix}I_B&0\\0&0\end{pmatrix},$$

and we denote $P_1CQ_1=\begin{pmatrix}C_1&C_2\\C_3&C_4\end{pmatrix}$, where the orders of matrices C_1, C_2, C_3, C_4 are determined by the blocks in P_1BQ_1, respectively.

Since

$$rk(B\ \ C)=rk[P_1(B\ \ C)\begin{pmatrix}Q_1&0\\0&Q_1\end{pmatrix}]=rk\begin{pmatrix}I_B&0&C_1&C_2\\0&0&C_3&C_4\end{pmatrix}=rk(B)+rk(C),$$

we know that $rk(C_3\ \ C_4)=rk(C)$, and there exist nonsingular matrix P_2 such that

$$P_2P_1BQ_1=\begin{pmatrix}I_B&0\\0&0\end{pmatrix}\quad\text{and}\quad P_2P_1CQ_1=\begin{pmatrix}0&0\\C_3&C_4\end{pmatrix}.$$

Since

$$rk\begin{pmatrix}B\\C\end{pmatrix}=rk[\begin{pmatrix}P_2P_1&0\\0&P_2P_1\end{pmatrix}\begin{pmatrix}B\\C\end{pmatrix}Q_1]=rk\begin{pmatrix}I_B&0\\0&0\\0&0\\C_3&C_4\end{pmatrix}=rk(B)+rk(C),$$

it holds that $rk(C_4)=rk(C)$ and there exists nonsingular matrix Q_2 such that

$$P_2P_1BQ_1Q_2=\begin{pmatrix}I_B&0\\0&0\end{pmatrix}\quad\text{and}\quad P_2P_1CQ_1Q_2=\begin{pmatrix}0&0\\0&C_4\end{pmatrix}.$$

Furthermore, there exist nonsingular matrices P_3 and Q_3 of order $n-rk(B)$ such that

$$P_3C_4Q_3=\begin{pmatrix}0&0\\0&I_C\end{pmatrix}$$

and

$$\begin{pmatrix} I_B & 0 \\ 0 & P_3 \end{pmatrix} P_2P_1BQ_1Q_2 \begin{pmatrix} I_B & 0 \\ 0 & Q_3 \end{pmatrix} = \begin{pmatrix} I_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} I_B & 0 \\ 0 & P_3 \end{pmatrix} P_2P_1CQ_1Q_2 \begin{pmatrix} I_B & 0 \\ 0 & Q_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_C \end{pmatrix}.$$

Let $P = \begin{pmatrix} I_B & 0 \\ 0 & P_3 \end{pmatrix} P_2P_1$ and $Q = Q_1Q_2 \begin{pmatrix} I_B & 0 \\ 0 & Q_3 \end{pmatrix}$, then

$$PBQ = \begin{pmatrix} I_B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad PCQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_C \end{pmatrix}.$$

Finally, the equivalence between (7), (9) and (10) can easily be seen by Theorem 2.1. This completes the proof.

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块阵广义逆的块独立性的一些等价条件

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摘 要: 基于 [1] 中关于块阵广义逆的块独立性的讨论，给出了两个同阶矩阵关于 {1,2}-逆具有块独立性的一些等价条件.