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Indecomposable Decomposations of R-Quasi-continuous Modules *

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Abstract: It is shown that if M is an R-quasi-continuous left R-module and R satisfies ACC on left ideals of the form $l(m)$, $m \in M$, then M is a direct sum of uniform submodules.

Key words: R-quasi-continuous module; R-extending module; uniform module.

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It is well-known that any injective left R-module M over a noetherian ring R is a direct sum of injective uniform submodules. Muller and Rizvi in [1] showed that a ring R is left noetherian if and only if every continuous left R-module has an indecomposable decomposition. This result was generalized to extending modules by Okado [2] and to R-continuous modules by Lopez-Permouth, Oshiro and Rizvi [3]. Thus all extending left R-modules and all R-continuous left R-modules over a left noetherian ring R are the direct sum of uniform submodules. On the other hand, a result in [4] states that if M is an extending left R-module and R satisfies ACC on left ideals of the form $l(m)$, $m \in M$, then M is a direct sum of uniform submodules. In this paper we will show that if M is an R-quasi-continuous left R-module and R satisfies ACC on left ideals of the form $l(m)$, $m \in M$, then M is a direct sum of uniform submodules.

All rings considered here are associative with identity. If $K < L$ is an essential submodule of L we write $K \lhd L$.

Let M, N be left R-modules. Define the family

$$\mathcal{A}(N,M) = \{A \subseteq M \mid \exists X \subseteq N, \exists f \in \mathrm{Hom}(X,M), f(X) \lhd A\}.$$

Considering the properties

$\mathcal{A}(N,M)$-(C_1): For all $A \in \mathcal{A}(N,M)$, $\exists A^* | M$, such that $A \lhd A^*$.

$\mathcal{A}(N,M)$-(C_2): For all $A \in \mathcal{A}(N,M)$, if $X|M$ is such that $A \cong X$, then $A|M$.

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$\mathcal{A}(N,M)$-(C_3): For all $A \in \mathcal{A}(N,M)$ and $X|M$, if $A|M$ and $A \cap X = 0$ then $A \oplus X|M$.

According to [3], M is said to be N-extending, N-quasi-continuous or N-continuous, respectively, if M satisfies $\mathcal{A}(N,M)$-(C_1), $\mathcal{A}(N,M)$-(C_1) and $\mathcal{A}(N,M)$-(C_3), $\mathcal{A}(N,M)$-(C_1) and $\mathcal{A}(N,M)$-(C_2).

Lemma 1 *Condition $\mathcal{A}(N,M)$-(C_i) $(i = 1,2,3)$ is inherited by direct summands of M.*

In [4], it was proved that if M is an extending R-module and R satisfies ACC on left ideals of the form $l(m)$, $m \in M$, then M is a direct sum of uniform submodules. Here we have

Theorem 2 *Let M be an R-quasi-continuous left R-module and let R satisfies ACC on left ideals of the form $l(m)$, $m \in M$. Then M is a direct sum of uniform submodules.*

Proof Let m be a non-zero element of M such that $l(m)$ is maximal in $\{l(x)|x \in M, x \neq 0\}$. Clearly $Rm \in \mathcal{A}(R,M)$. Thus there exists a direct summand L such that Rm is essential in L. Suppose that L is not indecomposable. Then there exist non-zero submodules L_1 and L_2 of L such that $L = L_1 \oplus L_2$. There exist $m_i \in L_i$ $(i = 1,2)$ such that $m = m_1 + m_2$. If $m_1 = 0$, then $m = m_2 \in L_2$, and thus $Rm \leq L_2$, which implies that $Rm \cap L_1 = 0$. Thus $L_1 = 0$, a contradiction. Thus $m_1 \neq 0$. Clearly $l(m) \leq l(m_1)$. Hence $l(m) = l(m_1)$, by the choice of m. Similarly $m_2 \neq 0$ and $l(m) = l(m_2)$. Since $Rm_1 \neq 0$ it follows that $Rm_1 \cap Rm \neq 0$. Thus there exist $r_1, r_2 \in R$ such that

$$0 \neq r_1 m_1 = r_2 m = r_2 m_1 + r_2 m_2.$$

Hence $r_2 m_2 = 0$, and thus $r_2 \in l(m_2)$ but $r_2 \notin l(m_1)$, a contradiction. Thus L is indecomposable.

By Lemma 1, L is R-extending. For any $0 \neq a \in L$, since $Ra \in \mathcal{A}(R,L)$, there exists a direct summand K of L such that Ra is essential in K. Now $K = L$, and thus Ra is essential in L. This means that L is uniform.

Consider a family of cyclic submodules $\{Rm_i|i \in I\}$ such that $\oplus_{i\in I} Rm_i$ is essential in M. Clearly, $Rm_i \in \mathcal{A}(R,M)$, for each $i \in I$. The R-quasi-continuity of M yields that there exist direct summands L_i $(i \in I)$ of M such that Rm_i is essential in L_i for each $i \in I$. Clearly $\oplus_{i\in I} L_i$ is essential in M, while by the R-quasi-continuity of M, it follows that $\oplus_{i\in F} L_i$ is a direct summand of M for every finite subset F of I. We will show that $M = \oplus_{i\in I} L_i$.

Suppose that $M \neq \oplus_{i\in I} L_i$. Select $m \in M - \oplus_{i\in I} L_i$, such that $l(m)$ is maximal in $\{l(x)|x \in M - \oplus_{i\in I} L_i\}$. Now, there exists $r \in R$ such that $0 \neq rm \in \oplus_{i\in I} L_i$ since $M = \oplus_{i\in I} L_i$ is essential in M. Then $0 \neq rm \in \oplus_{i\in F} L_i$, for some finite subset F of I. We know that

$$M = (\oplus_{i\in F} L_i) \oplus X$$

for some $X \leq M$. Write $m = y + x$, where $x \in X$ and $y \in \oplus_{i\in F} L_i$. It is easy to see that $l(m) \leq l(x)$. Since $m \notin \oplus_{i\in I} L_i$, it follows that $x \notin \oplus_{i\in I} L_i$. The maximality of $l(m)$ then yields that $l(m) = l(x)$. Now since $rx = rm - ry \in X \cap \oplus_{i\in F} L_i = 0$, then $rm = 0$, a contradiction. Hence $M = \oplus_{i\in I} L_i$.

Now, since every L_i is uniform, then M is a direct sum of uniform submodules.

A left R-module M is called locally noetherian if every finitely generated submodule is noetherian.

Corollary 3 *Any locally noetherian R-quasi-continuous left R-module is a direct sum of uniform submodules.*

Proof Let M be a locally noetherian R-quasi-continuous left R-module. For any $m \in M$, $R/l(m) \cong Rm$. So $R/l(m)$ is a noetherian left R-module. It follows that R satisfies ACC on left ideals of the form $l(x)$, $x \in M$. Now the result follows from Theorem 2.

Corollary 4 *Let M be a non-singular R-quasi-continuous left R-module. Then M is a direct sum of uniform submodules if and only if R satisfies ACC on left ideals of the form $l(m), m \in M$.*

Proof The sufficiency is clear by Theorem 2.

Conversely, suppose that $M = \oplus_{i \in I} M_i$, where M_i is a uniform submodule of M for each $i \in I$. By analogy with the proof of [4, Corollary 8.4], we can show that R satisfies ACC on left ideals of the form $l(m)$, $m \in M$.

The following example shows that R-extending modules may not be extending.

Example 5 Let $R = \mathbf{Z}$ and $M = \oplus_{i \in I} R$ be any free R-module of infinite rank. It follows from [5] that M is R-extending. But M is not extending by [6].

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R- 拟连续模的不可分分解

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摘 要: 设 R 是环， M 是 R- 拟连续左 R- 模. 如果 R 关于形如 $l(m), m \in M$ 的左理想满足升链条件，则 M 可写成一致子模的直和.