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On the Panfactorical Property of Cayley Graphs *

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Abstract: A k-regular graph is called panfactorical, or even panfactorical respectively, if for every integer s, $1 \le s \le k$,there exists an s-factor, or $2[\frac{s}{2}]$-factor, in this graph. A criterion for checking an r-regular graph to be panfactorical or even panfactorical is established. It is proved that every Cayley graph of odd degree is panfactorical and every Cayley graph of even degree is even panfactorical by using this criterion. For a dihedral group, we prove that every connected Cayley graph on this group is panfactorial.

Key words: panfactorial; Cayley graph; finite group; factorization.

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1. Introduction

We only consider simple graphs and finite groups in this paper. For terminology and notations not defined here, we follow the references [1] [4] and [7].

Let G be a simple graph. Its vertex set is denoted by $V(G)$ and edge set by $E(G)$. The notation $(x, y) \in E(G)$ denotes that (x, y) is an edge of the graph G. A factor of the graph G is a spanning subgraph of G which has some given properties. If $G_1, G_2, \cdots, G_s$, $s \ge 2$, are edge-disjoint factors of the graph G such that

$$E(G) = \bigcup_{i=1}^{s} E(G_i),$$

then we write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_s = \bigoplus_{i=1}^{s} G_i$ and say that G is edge sums of graph G_i, $1 \le i \le s$. An r-regular spanning subgraph of graph G is called an r-factor of G. By this definition, each perfect matching is a 1-factor. Call a k-regular graph is panfactorical or even panfactorical if for every integer s, $1 \le s \le k$, there exists an s-factor or $2[s/2]$-factor in this graph . For example, for any integer n, K^{2n} and C^{2n} are panfactorical, but K^{2n+1} is only even panfactorical. For a given vertex-transitive graph with degree $\ge r$, it is hard

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to say whether G has an r-factoror not. Even if G has an r-factor, generally, we do not know how to construct it (see [2] and [6]).

For a graph G, Aut$G$ denotes its automorphism group and x^H the orbit of the vertex x under the acting of H, where $n \in V(G)$ and $H \prec \text{Aut}G$. Call G to be vertex-transitive if for $\forall x, y \in V(G)$, there exists $a \in \text{Aut}G$ such that $x^a = y$. For a finite group Γ and its subset $S, S^{-1} = S$,$1_\Gamma \notin S$, Cayley graph $G = \text{Cay}(\Gamma : S)$ is defined as follows

$V(\text{Cay}(\Gamma : S)) = \Gamma$;

$E(\text{Cay}(\Gamma : S)) = \{(g, sg) | g \in \Gamma, s \in S\}$.

The edges (g, sg) is colored by s, where, $g \in \Gamma, s \in S$. It is obvious that $G = \text{Cay}(\Gamma : S)$ is regular and vertex-transitive, and is connected iff S is a generating set of Γ.

For Cayley graphs, there is a well-known but difficult conjecture not been solved yet untill today which asserts that every connected Cayley graph with more than 2 vertices is hamiltonian (see [3] [8]). Since finding a hamiltonian cycle in Cayley graph is difficult(see [3] [8]), we turn our attention to the panfactorical property in Cayley graphs. Notice that a 2-factor in a graph G is a union of cycles and a hamiltonian cycle is a special 2-factor. The aim of this paper is to establish a factorization for Cayley graphs in which each factor is 1-factor or 2-factor and using this factorization to prove that every Cayley graph of odd degree is panfactorical, every Cayley of even degree is even panfactorical.

2. A condition for a regular graph to be panfactorical

For deciding a k-regular graph is panfactorical or not, it does not need to check whether there existing all the s-factors, $1 \le s \le k$, since a criterion can be easily established.

Lemma 2.1 ([4],theorem 8.10) *A nonempty graph G is 2-factorable iff G is $2n$-regular for some $n \ge 1$.*

Using Lemma 2.1, a result for even panfactorial property of graphs can be obtained as follows.

Theorem 2.1 *A k-regular graph G is even panfactorical iff $k \equiv 0(\text{mod}2)$.*

Proof It is obvious that each vertex has even degree in a even panfactorical graph. So we only need to construct every even factor in the graph with $k \equiv 0(\text{mod}2)$. According to Lemma 2.1, we know that

$$G = F_1^2 \oplus F_2^2 \oplus \cdots \oplus F_{\frac{k}{2}}^2,$$

where $F_i^2, 1 \le i \le \frac{k}{2}$, is a 2-factor in G. Therefore, for any even integer $2s, 1 \le s \le \frac{k}{2}$, the union of s 2-factors is a $2s$-factor in this graph. So G is even panfactorical. □

Theorem 2.2

i) A $(2k+1)$-regular graph is panfactorical iff there is a 1-factor in this graph.

ii) A $2k$-regular graph is panfactorical iff there are 1-factor,3-factor,$\cdots$, k-factor,in this graph if $k \equiv 1(\text{mod}2)$ or there are 1-factor,3-factor,$\cdots$, $(k-1)$-factor,in this graph if $k \equiv 0(\text{mod}2)$.

Proof The necessarity of i) and ii) is obvious. it only need to prove its sufficiency.

i) Let G be a $(2k+1)$-regular graph. If there is a 1-factor F^1 in G, then $G\backslash F^1$ is a $2k$-regular graph. According to Lemma 2.1, we get

$$G\backslash F^1 = F_1^2 \oplus F_2^2 \oplus \cdots \oplus F_k^2,$$

where, $F_i^2, 1 \leq i \leq k$, is 2-factors in $G\backslash F^1$.Therefore,

$$G = F^1 \oplus F_1^2 \oplus F_2^2 \oplus \cdots \oplus F_k.$$

Now we construct every s-factor in G, where $1 \leq s \leq 2k+1$.

Case 1 $s \equiv 0(\mathrm{mod}2)$

Let $s = 2m, 1 \leq m \leq k$, then the union of m 2-factors is an s-factor in the graph G.

Case 2 $s \equiv 1(\mathrm{mod}2)$

Let $s = 2m+1, 1 \leq m \leq k$, then the union of m 2-factoes and F_1 is an s-factor in the graph G.

Combining Case 1 with Case 2, we know i) is true.

ii) Let G be a $2k$-regular graph. Notice that if F^r is an r-factor in G, then $G\backslash F^r$ is a $(2k-r)$-regular subgraph of G. If there are 1-factor, 3-factor, $\cdots$,k-factor(if $k \equiv 1(\mathrm{mod}2)$) or 1-factor, 3-factor,$\cdots$, $(k-1)$-factor(if $k \equiv 0(mod2)$) in the graph G, then there is $(2l-1)$-factor in G, where $1 \leq l \leq k$.

By Theorem 2.1, G is even panfactorical. Whence, G is panfactorical. □

3. A factorization of Cayley graph

Lemma 3.1[1] *A vertex-transitive graph G is a Cayley graph iff there is a regular subgroup in* Aut$G$.

For a permutation group H acting on a set S, there is a naturally induced acting of H on every 2-elements subset of S such that $\{u,v\}^h = \{uh, vh\}$ for $u, v \in S$ and $h \in H$. Especially, for a graph G and $H \prec \mathrm{Aut}G$,there is an induced acting of H on $E(G)$. Now choose a graph G and assume that $H \prec \mathrm{Aut}G$ and H is a regular group in the following discussion.

Lemma 3.2[7] $\forall x \in V(G), x^H = V(G)$ *and* $H_x = 1_H$.

Lemma 3.3[7] *For* $\forall (x,y),(u,w) \in E(G)$, $(x,y)^H \bigcap (u,w)^H = \emptyset$ *or* $(x,y)^H = (u,w)^H$.

Lemma 3.4 *For* $\forall (x,y) \in E(G), |H_{(x,y)}| = 1$ *or* 2.

Proof Assume that $|H_{(x,y)}| \neq 1$. Since any element $h \in H_{(x,y)}$, we have $(x,y)^h = (x,y)$.That is $(x^h, y^h) = (x,y)$. So we have $x^h = x$ and $y^h = y$ or $x^h = y$ and $y^h = x$. For the first case we get $h = 1_H$ by Lemma 3.2. For the second case, we get $x^{h^2} = x$. Therefore, we know $h^2 = 1_H$.

Now if there exists an element $g \in H_{(x,y)}\backslash\{1_H, h\}$, then we get $x^g = y = x^h$ and $y^g = x = y^h$. whence we have $g = h$ by Lemma 3.2, a contradiction. So we must have $|H_{(x,y)}| = 2$. □

Lemmma 3.5 *For any* $(x,y) \in E(G)$, *if* $|H_{(x,y)}| = 1$, *then* $(x,y)^H$ *is a 2-factor.*

Proof Since

$$x^H = V(G) \subset V(G[(x,y)^H]) \subset V(G),$$

so

$$V(G[(x,y)^H]) = V(G).$$

Therefore, $(x,y)^H$ is a spanning subgraph of G.

Since H acting on $V(G)$ is transitive, there exists $h \in H$ such that $x^h = y$. It is obvious that $o(h)$ is finite and $o(h) \neq 2$. Otherwise, we have $|H_{(x,y)}| \geq 2$, a contradiction. Now $(x,y)^{\prec h \succ} = xx^h x^{h^2} \cdots x^{h^{o(h)-1}} x$ is a cycle in the graph G. Consider the right coset decomposition of H on $\prec h \succ$. Suppose $H = \bigcup\limits_{i=1}^{s} \prec h \succ a_i$ and $\prec h \succ a_i \bigcap \prec h \succ a_j = \emptyset$, if $i \neq j, a_1 = 1_H$.

Let $X = \{a_1, a_2, ..., a_s\}$. We know that for any $a, b \in X$,

$$(\prec h \succ a) \bigcap (\prec h \succ b) = \emptyset,$$

if $a \neq b$. Since $(x,y)^{\prec h \succ a} = ((x,y)^{\prec h \succ})^a$ and $(x,y)^{\prec h \succ b} = ((x,y)^{\prec h \succ})^b$ are also cycles, if

$$V(G[(x,y)^{\prec h \succ a}]) \bigcap V(G[(x,y)^{\prec h \succ b}]) \neq \emptyset$$

for some $a, b \in X, a \neq b$, then there must be two elements $f, g \in \prec h \succ$ such that $x^{fa} = x^{gb}$. According to Lemma 3.2, we get $fa = gb$, that is $ab^{-1} \in \prec h \succ$. So $\prec h \succ a = \prec h \succ b$ and $a = b$, contracdits to the assumption that $a \neq b$.

Therefore, we know that $(x,y)^H = \bigcup\limits_{a \in X} (x,y)^{\prec h \succ a}$ is a disjoint union of cycles. So $(x,y)^H$ is a 2-factor of the graph G. □

Lemma 3.6 *For any $(x,y) \in E(G)$, $(x,y)^H$ is a 1-factor if $|H_{(x,y)}| = 2$.*

Proof Simlar to the proof of Lemma 3.5, we know that $V(G[(x,y)^H]) = V(G)$ and $(x,y)^H$ is a spanning subgraph of the graph G.

Let $H_{(x,y)} = \{1_H, h\}$, where $x^h = y$ and $y^h = x$. Notice that $(x,y)^a = (x,y)$ for $a \in H_{(x,y)}$. Consider the coset decomposition of H on $H_{(x,y)}$, we know that

$$H = \bigcup_{i=1}^{t} H_{(x,y)} b_i,$$

where $H_{(x,y)} b_i \bigcap H_{(x,y)} b_j = \emptyset$ if $i \neq j, 1 \leq i, j \leq t$. Now let $L = \{H_{(x,y)} b_i, 1 \leq i \leq t\}$. We get the decomposition of $(x,y)^H$:

$$(x,y)^H = \bigcup_{b \in L} (x,y)^b.$$

Notice that if $b = H_{(x,y)} b_i \in L$, $(x,y)^b$ is an edge of G. Now if there exists two elements $c, d \in L, c = H_{(x,y)} f$ and $d = H_{(x,y)} g$, $f \neq g$ such that

$$V(G[(x,y)^c]) \bigcap V(G[(x,y)^d]) \neq \emptyset,$$

there must be $x^f = x^g$ or $x^f = y^g$. If $x^f = x^g$, we get $f = g$ by Lemma 3.2, contracdits to the assumption that $f \neq g$. If $x^f = y^g = x^{hg}$, where $h \in H_{(x,y)}$, we get $f = hg$ and $fg^{-1} \in H_{(x,y)}$, so $H_{(x,y)}f = H_{(x,y)}g$. According to the definition of L, we get $f = g$, also contradicts to the assumption that $f \neq g$. *Therefore, $(x,y)^H$ is a 1-factor of the graph G.* □

Now we can obtain a factorization for Cayley graph, which is useful in determining factors of Cayley graph.

Theorem 3.1 *Let G be a vertex-transitive graph and H be a regular subgroup of* Aut$G$. *Then for any chosen vertex $x, x \in V(G)$, there is a factorization for G such that*

$$G = \Big(\bigoplus_{y \in N_G(x), |H_{(x,y)}|=1} (x,y)^H\Big) \bigoplus \Big(\bigoplus_{y \in N_G(x), |H_{(x,y)}|=2} (x,y)^H\Big), \qquad (3.1)$$

where $(x,y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and is a 1-factor if $|H_{(x,y)}| = 2$.

Proof For any chosen vertex $x, x \in V(G)$, according to Lemma 3.2-Lemma 3.4, we know that

$$G = \Big(\bigoplus_{y \in N_G(x), |H_{(x,y)}|=1} (x,y)^H\Big) \bigoplus \Big(\bigoplus_{y \in N_G(x), |H_{(x,y)}|=2} (x,y)^H\Big).$$

By Lemma 3.5 and Lemma 3.6, we know that $(x,y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and is an 1-factor if $|H_{(x,y)}| = 2$. Whence, we get the factorization (3.1) and the proof is complete. □

Now for a Cayley graph Cay$(\Gamma : S)$, we can always choose the vertex $x = 1_\Gamma$ and H is the right regular transformation group on Γ. We restate Theorem 3.1 as follows.

Theorem 3.2 *Let Γ be a finite group with its a subset $S, S^{-1} = S$, $1_\Gamma \notin S$ and H is the right transformation group on Γ. Then there is a factorization for the Cayley graph* Cay$(\Gamma : S)$ *such that*

$$G = \Big(\bigoplus_{s \in S, s^2 \neq 1_\Gamma} (1_\Gamma, s)^H\Big) \bigoplus \Big(\bigoplus_{s \in S, s^2 = 1_\Gamma} (1_\Gamma, s)^H\Big), \qquad (3.2)$$

where $(1_\Gamma, s)^H$ is a 2-factor if $s^2 \neq 1_\Gamma$ and is a 1-factor if $s^2 = 1_\Gamma$.

Proof For any $h \in H_{(1_\Gamma, s)}$, if $h \neq 1_\Gamma$, then we have $1_\Gamma h = s$ and $sh = 1_\Gamma$,that is $s^2 = 1_\Gamma$. According to Theorem 3.1, we get the factorization (3.2) for the Cayley graph Cay$(\Gamma : S)$. □

We give some examples to show the extremal cases in (3.2).

Example 3.1 Let $\Gamma = \prec a_1, a_2, \cdots, a_n | a_1^2 = a_2^2 = \cdots = a_n^2 = 1_\Gamma >$ be an Abelian group. Choose $S = \{a_1, a_2, \cdots, a_n\}$. Then every element in S is a convolution. According to Theorem 3.2, we get the 1-factorization for Cay$(\Gamma : S)$

$$\mathrm{Cay}(\Gamma : S) = \bigoplus_{s \in S} (1_\Gamma, s)^H.$$

Example 3.2 For any integer $m, n, m \geq 1, n \geq 1$, let $\Gamma = \prec a, b | a^{2m+1} = 1, b^{2n+1} = 1 >$ be an Abelian group. Choose $S = \{a, b, a^{2m}, b^{2n}\}$. Then the Cayley graph Cay$(\Gamma : S)$ is a 4-regular graph. Now since $|\Gamma| = (2m+1)(2n+1) \equiv 1(\mathrm{mod}2)$, there does not exists 1-factor in the Cayley graph Cay$(\Gamma : S)$, so we have a 2-factorization for Cay$(\Gamma : S)$

$$\mathrm{Cay}(\Gamma : S) = \bigoplus_{s \in S} (1_\Gamma, s)^H.$$

4. The factorial and panfactorial property of Cayley graphs

According to Theorem 3.2, we can immediately get the factorical property of Cayley graphs.

Theorem 4.1 *Let $G = \mathrm{Cay}(\Gamma : S)$ be a Cayley graph. Then*

i) Every G with $|S| \geq 2$ has a 2-factor.

ii) Every G has a 1*-factor if* $|S| \equiv 1(\mathrm{mod}2)$,*and especially,every Cayley graph of degree* 3 *is a union of a* 1*-factor and a* 2*-factor.*

iii) Every G with $|S| \equiv 0(\mathrm{mod}2)$ has a 2-factorization.

iv) Every G with $|\Gamma| \equiv 1(\mathrm{mod}2)$ has a 2-factorization;

v) G has every k-factor, where $1 \leq k \leq 3$,if $|S| \geq 4, |\Gamma| \equiv 0(\mathrm{mod}2)$ and has a 1-factor if $|S| \neq 2, |\Gamma| \equiv 0(\mathrm{mod}2)$.

Proof Assume that H is the right regular transformation group on Γ.

i) According to Theorem 3.2, if every $(1_\Gamma, s)^H, s \in S$, is not a 2-factor, then there are two 1-factors $(1_\Gamma, s_1)^H, (1_\Gamma, s_2)^H$ at least,where $s_1, s_2 \in S$ and $s_1^2 = s_2^2 = 1_\Gamma$.The union $(1_\Gamma, s_1)^H \bigcup (1_\Gamma, s_2)^H$ forms a 2-factor of the graph G.

ii) If $s \in S$ and $s^2 \neq 1_\Gamma$, then since $S = S^{-1}$, we know that $s^{-1} \neq s$ and $s^{-1} \in S$, so $\{s, s^{-1}\} \subset S$, that is, the non-convolution elements appear in pairs in S. Whence we get that there is a convolution r in S. Therefore, $(1_\Gamma, r)^H$ is an 1-factor and this result can be obtained from Theorem 3.2.

iii) According to Theorem 3.2, if there does not exist $s, s \in S$ such that $s^2 = 1_\Gamma$, then (3.2) is a 2-factorization for Cay$(\Gamma : S)$. Otherwise, there is a convolution in S. We prove that there are even convolutions in S. In fact, if $s^2 \neq 1_\Gamma$, then $\{s, s^{-1}\} \subset S, s \neq s^{-1}$.Since $|S| \equiv 0(\mathrm{mod}2)$, so we get that there are even convolutions in S. Since the union of two 1-factors is a 2-factor, the conclusion is obtained.

iv) Since every graph with odd order has not 1-factor, so for any $s, s \in S, s^2 \neq 1_\Gamma$. We get the factorization

$$G = \bigoplus_{s \in S} (1_\Gamma, s)^H,$$

where $(1_\Gamma, s)^H$ is a 2-factor in G for each $s, s \in S$.

v) If there exists a convolution in S, the conclusion is obvious. So we assume that there is not a convolution .If there exists $s, s \in S$ such that $o(s) \equiv 0(\mathrm{mod}2)$,then $(1_\Gamma, s)^H$ be a union of cycles with even order. We can choose an 1-factor from the subgraph $(1_\Gamma, s)^H$ for G and get k-factors, $1 \leq k \leq 3$. Now if $\forall s \in S, o(s) \equiv 1(\mathrm{mod}2)$,since $(1_\Gamma, s)^H$ is a

2-factor in G and $|G| = |\Gamma| \equiv 0(\mathrm{mod}2)$, simlar to the proof of Lemma 3.5 we get that

$$(1_\Gamma, s)^H = \bigcup_{a \in X} (1_\Gamma, s)^{\prec s \succ a} = \{(1_\Gamma, s)^{\prec s \succ a}, 1 \leq i \leq 2k, a_1 = 1_\Gamma\}.$$

Since $|S| \geq 4$,there exists $s' \in S$ such that $s' \in \prec s \succ$. Not loss of generality, assume that $s' \in \prec s \succ a_2$, then

$$F = \{(1_\Gamma, s')^{\prec s \succ a}, i \equiv 1(\mathrm{mod}2), 1 \leq i \leq 2k\}.$$

Forms a 1-factor and $F \bigcup (1_\Gamma, s)^H$ is a 3-factor in G. So the conclusion is true. □

Joseph Zaks conjectured that for all $d \geq 4$ or $d = 2$ and $q \geq 3$, the Parsons graph $T_b(d,q)$ has a 1-factor. This conjecture is proved in [5].But since every Parsons graph $T_b(d,q)$ is a Cayley graph $\mathrm{Cay}(\Gamma : S)$, where $\Gamma = SL(d,q)$ and $S = S_b(d,q) = \{A \in SL(d,q) | det(A - I) = b\}$, and $|\Gamma| \equiv 0(\mathrm{mod}2)$, so from Theorem 4.1.v), we can also get this conjecture is true.

Now we turn our attention to the panfactorical property of Cayley graphs.

Theorem 4.2

i) Every Cayley graph G of odd degree is panfactorical.

ii) Every Cayley graph G of even degree is even panfactorical.

Proof i) Since the union of 2-factors is a even regular graph, now if G is a $(2k+1)$-regular graph,$k \geq 1$, there must be a 1-factor in G according to Theorem3.2. Therefore, we get that G is panfactorical by Theorem 2.2.

ii) This result is a corollary of Theorem 2.1. □

For Cayley graphs on dihedral group, we have the following more strong result for their panfactorial property.

Theorem 4.3 *Every connected Cayley graph on a dihedral group is panfactorial.*

Proof Let $D_n = \prec a, b | a^n = b^2 = 1, aba = b \succ = \{1, a, a^2, ... a^{n-1}, b, ba, ..., ba^{n-1}\}$ be a dihedral group. Similar to the proofs of Theorem 4.1 ii) and Theorem 3.2, we only need to prove that every generating set for D_n contains a convolution. Notice that for any integer $i, 0 \leq i \leq n-1, ba^i$ is a convolution since

$$\begin{aligned}(ba^{i-1})^2 &= ba^{i-1}(aba)a^{i-1} = ba^{i-1}ba^{i-1} = ba^{i-2}ba^{i-2} \\ &= \cdots\cdots\cdots = b^2 = 1.\end{aligned}$$

Because any generating set must has a $ba^i, 0 \leq i \leq n-1$, type element, whence S contains a convolution. □

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Cayley 图的泛因子性质

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摘 要: 一个 k- 正则图若满足对任意正整数 $s, 1 \leq s \leq k$, 均存在一个 s- 因子或一个 $2[\frac{s}{2}]$ 因子，则称其有泛因子或偶泛因子性质. 本文证明了每个奇度 Cayley 图是泛因子的，每个偶度 Cayley 图是偶泛因子的. 同时证明了二面体群上的每个 Cayley 图均是泛因子的.