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Completely Positive Realizations of a Cycle *

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Abstract: An $n \times n$ real matrix A is called doubly nonnegative, if A is entrywise nonnegative and semidefinite positive as well. A is called completely positive if A can be factored as $A = BB^t$,where B is some nonnegative $n \times m$ matrix. The smallest such number m is called the factorization index (or CP-rank) of A. This paper presents a criteria for a doubly nonnegative matrix realization of a cycle to be completely positive, which is strightforward and effective.

Key words: doubly nonnegative matrix; completely positive graph; cycle; factorization index.

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1. Introduction

Completely positive matrices are important in the study of block designs in combinatorial analysis, and have applications in establishing economic model [8].

Recall that an $n \times n$ matrix A is said to be completely positive, denoted by $A \in CP_n$, if there exist m nonnegative column vectors $b_1, \cdots, b_m$ such that

$$A = b_1 b_1^t + \cdots + b_m b_m^t,$$

where t denotes transpose. The smallest such number m is called the factorization index of A and denoted by $\phi(A)$. An $n \times n$ nonnegative matrix A is called doubly nonnegative, denoted by $A \in DP_n$, if it is semidefinite positive. It is known that

Lemma 1[1,3] *$DP_n = CP_n$ for $n \leq 4$.*

But for $n > 4$, CP_n is a proper subset of DP_n (see [1,3]).

we denote $A(l)$ the submatrix of A obtained by deleted the lth row and column of A. Let $E_{rs} = (e_{ij})$ denotes an $n \times n$ matrix,where $e_{rs} = 1$,otherwise, $e_{ij} = 0$. For a real symmetric matrix A, the graph $G(A) = (V, E)$ of A is defined as: $V = \{1, \cdots, n\}$ and

$$E = \{\{i, j\} : \ i \neq j, a_{ij} \neq 0, \ i, j = 1, \cdots, n\}.$$

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For any vertex $l \in V$, let $N(l)$ denote the set of all neighbors of l in G, i.e.,

$$N(l) = \{i : (i, l) \in E, i \neq l\}.$$

By a doubly nonnegative realization of a graph G, we mean a matrix $A \in DP_n$ for which $G(A) = G$. The set of all such matrices is denoted by Λ_G(see [11]). G is called completely positive (abbrev. *cp*)if $A \in CP_n$ for any $A \in \Lambda_G$. It is shown in [3,4,5] that

Lemma 2 *A graph G is cp if and only if G does not contain an odd cycle of length greater than 4.*

From lemma 2, we know that if G is acyclic(i.e., without any cycle) or G is an even cycle, then G is *cp*.

2. A necessary and sufficient condition

Let $G_1 \cong C_{2k+1}$(see [11]), i.e., G_1 is a cycle with length $2k+1$ where $k \geq 2$. By [3], we know that there are some(in fact, many)matrices in Λ_{G_1} which are non-*cp* and we would like to determine which matrices are *cp*.

It is easy to find that for any $A \in \Lambda_{G_1}$, A is permutation similar to a matrix of the form

$$\begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}, \tag{1}$$

where $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$. Hence, we assume that A is of form (1) from now on.

We have shown the following interesting result in [11]:

Theorem 1 *Let $A \in DP_n$ be of the form (1) where $a_{ij} \neq 0$ if and only if $j = i-1, i, i+1 (\mathrm{mod}\ n)$. Then A is in CP_n if and only if there exist two positive numbers a, b such that*

(i) $ab = a_{1n}$.

(ii) $H = A - (a^2 E_{11} + b^2 E_{nn} + a_{1n} E_{1n} + a_{n1} E_{n1}) \in DP_n$.

But it is not practical to use Theorem 1 to determine which matrices in Λ_{G_1} are *cp*.

The following interesting result gives a necessary and sufficient condition for any matrix $A \in \Lambda_{G_1}$ to be *cp*, which is proved to be more effective and convenient for us to use than any other one.

Theorem 2 *Let $A \in DP_n$ and that $G(A) \cong C_n$ with n an odd number large than 4. Then $A \in CP_n$ if and only if* $\det A \geq 4A_{C_{|n|}}$ *($A_{C_{|n|}}$ denotes the weight of the cycle $G(A)$).*

Proof Suppose $A \in CP_n$ be in the form (1). We may assume that A is irreducible and $a_{11} = a_{22} = \cdots = a_{nn} = 1$;Otherwise, we substitute the matrix A with $D^{-1/2}AD^{-1/2}$,where $D = diag(a_{11}, \cdots, a_{nn})$(note that $a_{ii} > 0$ for all i by the irreducibility of A). Then A can be factored as $A = BB^t$ where

$$B = \begin{bmatrix} b_{11} & b_{12} & & & \\ & b_{22} & b_{23} & & \\ & & \ddots & \ddots & \\ & & & & b_{n-1,n} \\ b_{n1} & & & & b_{n,n} \end{bmatrix}, \tag{2}$$

where B is a nonnegative matrix of order n, with $b_{ij} > 0$ for $j = i, i+1, i = 1, 2, \cdots, n(n+1 \equiv 1)$. Hence $\det A = (\det B)^2$. While

$$\begin{aligned}(\det B)^2 &= (b_{11}b_{22}\cdots b_{nn} + b_{12}b_{23}\cdots b_{n-1,n}b_{n1})^2 \\ &\geq 4b_{11}b_{22}\cdots b_{nn}b_{12}b_{23}\cdots b_{n-1,n}b_{n1} \\ &= 4a_{12}a_{23}\cdots a_{n-1,n}a_{n1},\end{aligned}$$

that is, $\det A \geq 4A_{|C_n|}$.

Conversely, suppose $\det A \geq 4A_{|C_n|}$. Partition A as the form

$$\begin{bmatrix} A_{11} & \alpha \\ \alpha^t & 1 \end{bmatrix}, \tag{3}$$

where $\alpha = (a_{1n}, 0, \cdots, 0, \cdots, a_{n-1,n})^t \in R^{n-1}$. It is easy to see that

$$\det A = \det(A_{11} - \alpha\alpha^t) = \det H + 4a_{12}a_{23}\cdots a_{n-1,n}a_{n1}. \tag{4}$$

Here H is the following symmetric nonnegative matrix of order $n-1$,

$$\begin{bmatrix} 1 - a_{1n}^2 & a_{12} & & & a_{1n}a_{n-1,n} \\ a_{21} & 1 & a_{23} & & \\ & a_{32} & a_{33} & \ddots & \\ & & \ddots & \ddots & \\ a_{1n}a_{n-1,n} & & & a_{n-1,n-2} & 1 - a_{n-1,n}^2 \end{bmatrix}, \tag{5}$$

from the inequality $\det A \geq 4A_{|C_n|}$ and (4),$\det H \geq 0$. Meanwhile, $A_{11} - \alpha\alpha^t$ is semidefinite positive by the double nonnegativity of A. Therefore $H \in DP_{n-1}$. But $G(H) \equiv C_{n-1}(n-1 = 2k)$ is an even cycle - a completely positive graph. So $H \in CP_{n-1}$. Moreover, H has a factorization $H = B_1 B_1^t$ with

$$B_1 = \begin{bmatrix} b_{11} & b_{12} & & & \\ & b_{22} & b_{23} & & \\ & & \ddots & \ddots & \\ & & & & b_{n-2,n-1} \\ b_{n-1,1} & & & & b_{n-1,n-1} \end{bmatrix}.$$

Here B_1 is a nonnegative matrix of order $n-1$. Put

$$b'_{11} = \sqrt{b_{11}^2 + a_{1n}^2}, \qquad b_{n-1,n} = \sqrt{b_{n-1,1}^2 + a_{n-1,n}^2},$$

$$b_{n1} = \frac{a_{1n}}{b'_{11}}, \qquad b_{nn} = \frac{a_{n-1,n}}{b_{n-1,n}},$$

and

$$B = \begin{bmatrix} b'_{11} & b_{12} & & & \\ & b_{22} & b_{23} & & \\ & & \ddots & \ddots & \\ & & & & b_{n-1,n} \\ b_{n1} & & & & b_{nn} \end{bmatrix}$$

(B is of order n) and denote $P = BB^t = (p_{ij})$. Comparing A with P, we have $p_{ij} = a_{ij}$ for all $(i,j) \neq (n,n)$. Next we want to prove $p_{nn} = a_{nn} = 1$. Since $b_{11}b_{n-1,1} = a_{1n}a_{n-1,n}$,

$$\begin{aligned} p_{nn} = b_{n1}^2 = b_{nn}^2 &= \frac{a_{1n}^2}{{b'_{11}}^2} + \frac{a_{n-1,n}^2}{b_{n-1,n}^2} \\ &= \frac{a_{1n}^2}{b_{11}^2 + a_{1n}^2} + \frac{a_{n-1,n}^2}{b_{n-1,1}^2 + a_{n-1,n}^2} \\ &= \frac{a_{1n}^2 a_{n-1,n}^2}{b_{11}^2 a_{n-1,n}^2 + b_{11}^2 b_{n-1,1}^2} + \frac{b_{11}^2 a_{n-1,n}^2}{b_{11}^2(b_{n-1,1}^2 + a_{n-1,n}^2)} \\ &= 1. \end{aligned}$$

Hence $A = BB^t$.

Generally, if we denote $\tilde{A} = D^{-1/2}AD^{-1/2} = (\tilde{a}_{ij})$, then $\tilde{A} \in CP_n$ if and only if $A \in CP_n$, and $\tilde{a}_{ij} = (a_{ii}a_{jj})^{-1/2}a_{ij}$. By using the above result to $\tilde{A}$ and noticing that $\det \tilde{A} = (a_{11}\cdots a_{nn})^{-1}\det A$, and $\tilde{A}_{|C_n|} = (a_{11}\cdots a_{nn})^{-1}A_{|C_n|}$, we obtain $A \in CP_n \rightleftharpoons \det A \geq 4A_{|C_n|}$.

Corollary 2 *Let $A \in CP_n$ and suppose that $G(A)$ is an odd cycle of length greater than 4. Then $\varphi(A) = n$.*

This result is immediate from the proof of Theorem 2.

Corollary 3 *Let $A \in DP_n$ be in form (1) with n an even number. Then $\det A \geq 4A_{|C_n|}$.*

Proof Since n is an even number, $G(A) \cong C_n$ is *cp*. Hence $A \in DP_n$ implies $A \in CP_n$. By Theorem 2, we get $\det A \geq 4A_{|C_n|}$.

We conclude the discussion by the following example which illustrate that the condition $\det A \geq 4A_{|C_n|}$ generally does not imply that $A \in DP_n$ even if A is in form (1).

Example Let A be the matrix

$$\begin{bmatrix} 1 & 0.0616715 & 0 & 0 & 0 & 0.762924 \\ 0.0616715 & 1 & 0.810464 & 0 & 0 & 0 \\ 0 & 0.810464 & 1 & 0.868167 & 0 & 0 \\ 0 & 0 & 0.868167 & 1 & 0.546131 & 0 \\ 0 & 0 & 0 & 0.546131 & 1 & 0.929494 \\ 0.762924 & 0 & 0 & 0 & 0.929494 & 1 \end{bmatrix}.$$

Then simple calculations yield

$$\det A = 0.107739, \ \det A - 4A_{|G|} = 0.0405173.$$

But A is not semidefinite positive, since $\det A(6) < 0$.

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关于圈的完全正矩阵实现

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摘 要: 本文给出了一个关联图为圈的非负、半正定矩阵 A 为完全正的一个充要条件. 我们还证明了这样的矩阵 A(当 A 为完全正时) 的分解指数即为 A 的阶数.