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Minimum Coverings of Complete Directed Graphs with Odd Size Circuits *

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Abstract: Let DK_v denote the symmetric complete directed graph with v vertices, the covering number $C(v,m)$ is a minimum number of covering DK_v by m-circuits. In this paper, $C(v,m)$ is determined for any fixed odd positive integer m and positive integer v, $m \le v \le m+6$.

Key words: m-circuits; covering number; complete directed graph.

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1. Introduction

Let DK_v denote the symmetric complete directed graph with v vertices, where any two distinct vertices x and y are joined by exactly two arcs (x,y) and (y,x). By an m-circuits we mean directed elementary circuit of length m. An m-circuits $\langle a_1, a_2, \cdots, a_m \rangle$ contains m arcs $(a_1,a_2),(a_2,a_3),\cdots,(a_{m-1},a_m),(a_m,a_1)$. Covering number $C(v,m)$ is minimum number of covering DK_v by m-circuits. Let $T(v,m) = \lceil \frac{v(v-1)}{m} \rceil$. It is easy to see that $C(v,m) \ge T(v,m)$. We denote by (v,m)-DCC the set of m-circuits whose union is DK_v. If $C(v,m) = T(v,m)$ for given v and m, then a minimum (v,m)-DCC is said to be optimal.

We remark that an optimal (v,m)-DCC is essentially an arc-disjoint decomposition of DK_v into m-circuits whenever $v(v-1) \equiv 0 \pmod m$. Many researchers have been involved in decomposing a DK_v into arc-disjoint m-circuits. For any positive integer $v \ge m$, F.E.Bennett and J.X yin[1] determined the values of $C(v,m)$ for m=3,4. Furthermore, for any fixed even integer m J.X.Yin[2] reduce the determination of the values of $C(v,m)$ to the case where $m+3 \le v \le 2m-2$, and he proved the $C(v,m)$=$T(v,m)$ for m=6,8 and 10 except $C(6,6)$=$T(6,6)+1$. Q.D.Kang and Z.H.Liang [3,4,5,6] reduced the determination of the value of $C(v,m)$ to the case where $m+6 \le v \le 2m-4$ for any fixed even integer $m \ge 4$. In particular, the values of $C(v,m)$ are completely determined for $m \le 17$.

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Let $m = p^r, pq, p^2q, p^3q$, where p and q are distinct odd primes, and $m \geq 19$. If there exist optimal (v,m)-DCC for $[m+5, 2m-4] \bigcup [2m+5, 3m-4]$, then there exist optimal (v,m)-DCC for $v \geq m$.

For convenience sake, we shall use the following notations

$$Z_n^* = Z_n \backslash \{0\},$$

$$[a,b] = \{x \in Z : a \leq x \leq b\},$$

$$[a,b]_k = \{x \in Z : a \leq x \leq b, \quad \text{and} \quad x \equiv a (\mathrm{mod} k)\},$$

where Z is the integer ring, $a, b, k \in Z$ and $k > 1$. The symbol $\lceil x \rceil (\lfloor x \rfloor)$ is the least (greatest) integer y satisfying $y \geq x (y \leq x)$.

2. Main result

Lemma *If $mT(v,m) - v(v-1) = 1$, then $C(v,m) \geq T(v,m) + 1$.*

Theorem 1 *For any odd integer $m \geq 19$, $C(m+5, m) = T(m+5, m)$.*

Construction Let the vertex set of DK_{m+5} be $Z_{m+4} \bigcup \{\infty\}$. Define

$$X(i) = \langle \infty, a_2, a_3, \cdots, a_m \rangle + i, i \in Z_{m+4},$$

where

$$a_{2i-1} = i - 1, i \in [2, \frac{m+1}{2}], a_{2i} = \begin{cases} m+3-i, & \text{if } i \in [1, \lfloor \frac{m+1}{4} \rfloor], \\ m+2-i, & \text{if } i \in [\lfloor \frac{m+5}{4} \rfloor, \frac{m-1}{2}]. \end{cases}$$

Case 1. When $m \geq 27$, we again increase 6 m-circuits:

$$A = \langle 0, 1, 2, ..., m-5, m-4, m-2, m, m+2 \rangle,$$
$$B = \langle 0, m+3, m+2, m+1, ..., 10, 9, 7, 5, 3, 1 \rangle,$$
$$C = \langle 0, (m+5)/2, 1, (m+7)/2, 2, (m+9)/2, ..., (m-5)/2, m, m+1, m+2, m+3 \rangle,$$
$$D = \langle 0, 2, 4, ..., m+3, 1, 3, 5, ..., m-6 \rangle,$$
$$E = \langle 1, m+3, m+1, m-1, ..., 2, 0, m+2, m, m-2, ..., 11 \rangle,$$
$$F = \langle m-6, m-4, m-3, m-2, m-1, m, \frac{m-3}{2}, m+1, \frac{m-1}{2}, m+2, \frac{m+1}{2},$$
$$m+3, \frac{m+3}{2}, 0, 11, 9, 8, 7, 6, 5, 4, 3, 2, 1, P \rangle,$$

where $P = [(m+9)/2, m-7]$ if $m = 27$; $P = [(m+9)/2, m-7] \bigcup [12, (m-5)/2]$ if $m > 27$. Obviously, P is a sequence with length $m-24$. Repeated arcs are $\langle m-6, 0, 11, 1, P \rangle$.

Case 2. When $m = 25$, A, B, C, D and E are the same as the Case 1. The F is modified

$$\langle 19, 21, 22, 23, 24, 25, 17, 26, 12, 27, 13, 28, 14, 0, 11, 9, 8, 7, 6, 5, 4, 3, 2, 1, 15 \rangle.$$

We need to break up two m-circuits $X(-2)$ and $X(-7)$, and to form two new m-circuits as follows:

$$\langle \infty, 25, 28, 24, 0, 23, 1, 11, 2, 21, 3, 20, 4, 18, 5, 17, 6, 16, 7, 15, 8, 14, 9, 13, 10 \rangle,$$

$\langle\infty, 20, 23, 19, 24, 18, 25, 11, 26, 16, 27, 15, 28, 13, 0, 12, 1, 22, 2, 10, 3, 9, 4, 8, 5\rangle$.

When m=19, 21 and 23, we can also obtain an optimal $(m+5, m)$-DCC.

Theorem 2 *For all odd integer $m \geq 17$, $C(m+6, m) = T(m+6, m)$, except for $C(37, 31) = T(37, 31) + 1$.*

Construction Let the vertex set of DK_{m+6} be $Z_{m+5} \bigcup \{\infty\}$. Define

$$X(i) = \langle\infty, a_2, a_3, \cdots, a_m\rangle + i, i \in Z_{m+5},$$

where the $a_{2i} = -3 - i$, $1 \leq i \leq (m-1)/2$, $a_{2i+1} = i - 1$, $1 \leq i \leq (m-1)/2$. The differences $a_{i+1} - a_i$ run over $Z^*_{m+5} \backslash \{\pm 1, \pm 2, \pm 3\}$.

Case 1. $m \equiv 0, 2 \pmod 3$ and $17 \leq m \leq 33$, define

$$\begin{aligned}
A =&\langle 8, 10, 12, \cdots, m+3, 0, 1, m+4, m+2, \cdots, 9, 7\rangle,\\
B =&\langle m-1, m-3, m-5, \cdots, 4, 2, 1, 3, 5, \cdots, m\rangle,\\
C =&\langle 0, 2, 3, 4, 5, 6, 7, 10, 13, 14, 15, \cdots, m\rangle,\\
D =&\langle 6, 9, 12, \cdots, m+4, 2, 5, \cdots, m+3, 1, 4, 7, P\rangle,\\
E =&\langle m+4, m+3, m+2, m+1, m, m-3, m-4, m-5, \cdots, a+1, a, a-3, a-4, \cdots, 3, 2, 0\rangle,\\
F =&\langle 0, m+2, \cdots, 1, m+3, m, Q, m+1, m-2, \cdots, 6, 3\rangle,\\
G =&\langle m, m+4, 1, m-3, 2, 4, 6, 8, 9, 10, 11, 12, 13, 16, 19, 22, P^*, S\rangle,\\
H =&\langle 7, 3, 1, 0, m+3, m+1, m-1, m-2, m-3, m-6, m-9, \cdots,\\
&a, a-1, b, a-2, a-3, a-6, \cdots, Q^*, R\rangle,
\end{aligned}$$

where $m \equiv 2 \pmod 3$,

$$a = 14, b = m+2;$$

when $m \equiv 0 \pmod 3$,

$$a = 12, b = m+4.$$

$P = (25, 28, \cdots, 0, 3)$, $Q = (m-18, m-21, \cdots, 2, m+4)$, $P^* = 25$, $Q^* = m - 18$, both S and R are $m-17$ sequence. We modify two m-circuits $X(-1)$, $X(-(m-9)/2)$. for $X(-1)$, we delete $m-3$ and insert $m+2$ in between $a_2 - 1$ and $a_3 - 1$. For $X(-(m-9)/2)$, we insert 5 in between 7 and 3. When $m \equiv 2 \pmod 3$, delete $m+2$; when $m \equiv 0 \pmod 3$, delete $m+4$.

Case 2. When $m \equiv 1 \pmod 3$ and $17 \leq m \leq 29$, there exists optimal $(m+6, m)$-DCC. When m=31, $C(37, 31) = T(37, 31) + 1$.

Case 3. When $m \equiv 0, 2 \pmod 3$ and $m \geq 35$, define

$$\begin{aligned}
A =&\langle m-3, m-5, m-7, \cdots, 2, 0, 1, 3, 5, \cdots, m\rangle,\\
B =&\langle 8, 9, 10, 12, 14, 16, \cdots, m+3, 0, m+4, m+2, m, \cdots, 9, 7\rangle,\\
C =&\langle 0, 2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 16, 17, 18, 19, \cdots, m+3, m+4\rangle,\\
D =&\langle m+4, m+3, m+2, m+1, m, m-1, m-2, m-3, m-6, m-9, m-10, m-11, \cdots, 2, 1\rangle\\
E =&\langle 25, 28, 31, \cdots, m+3, 1, 4, 7\rangle,\\
F =&\langle m-18, m-21, m-24, \cdots, 7, 4, 1, m+3, m\rangle,\\
G =&\langle m, m+2, m+4, 1, 2, 4, 6, 10, 13, 14, 15, 0, 16, 19, 22, 25, 7, 5, 3, m+3, m+1, m-1,\\
&m-3, m-4, 11, m-5, 12, m-6, m-7, m-8, m-9, m-12, m-15, m-18, P\rangle,
\end{aligned}$$

where P is an $(m-34)$-sequence. In $X(3)$, delete 11 and 12,and insert $(1,0)$ in between 3 and $m+3$. For $X(10)$, delete 0, and insert 7 in between 6 and 10.

Case 4. When $m \equiv 1 \pmod 3$ and $m \geq 37$, define

$$\begin{aligned}
A =&\langle 12,15,18,\cdots,m+2,0,2,5,\cdots,m+1,m+4,1,m+3,m,m-3,\cdots,13,10\rangle,\\
B =&\langle m-4,m-7,\cdots,3,0,m+4,m+1,\cdots,5,2,1,4,7,\cdots,m-9,m-6\rangle,\\
C =&\langle 0,1,2,3,4,5,6,7,8,9,10,11,12,14,16,18,20,22,23,24,\cdots,m+4\rangle,\\
D =&\langle m+4,m+3,\cdots,m-6,m-7,m-9,m-11,m-13,m-14,m-15,\cdots,4,3,2\rangle,\\
E =&\langle 22,24,26,\cdots,m+3,0,m+2,m,m-2,\cdots,3,1,m+4,2,4,6,8,10\rangle,\\
F =&\langle m-13,m-15,\cdots,6,4,1,3,5,\cdots,m-6,m-3,m,m+2,m+4,2,0,m+3,\\
&m+1,m-1,m-4\rangle,\\
G =&\langle m-4,m-2,m,m+3,1,0,3,6,9,12,13,14,15,16,17,18,19,20,21,22,m+4,\\
&m+2,m-1,m-3,m-5,m-7,m-8,m-9,m-10,m-11,m-12,\\
&m-13,10,7,4,2,P\rangle,
\end{aligned}$$

where P is an $(m-36)$-sequence.

Proof First

$$T(m+6,m)=\lceil\frac{(m+6)(m+5)}{m}\rceil=\begin{cases} m+13, & \text{if } 17\leq m\leq 29,\\ m+12, & \text{if } m\geq 31.\end{cases}$$

The arcs of differences in $Z_{m+5}\backslash\{0,\pm1,\pm2,\pm3\}$ and arcs (∞,i), (i,∞), $i\in Z_{m+5}$ are contained in $X(i)$, $i\in Z_{m+5}$. Thus we only need to prove that other arcs in DK_{m+6} are contained in the new constructed m-circuits in Case 1-4.

In Case 1, there are $m+13$ m-circuits in this construction, repeated arcs are

$$\langle 7,P^*,S,m,Q^*,R\rangle.$$

Therefore, $C(m+6,m)=T(m+6,m)$. In case 2, it is easy to obtain that

$$C(m+6,m)=T(m+6,m),$$

except for $C(37,31)=T(37,31)+1$.

In Case 3, distribution of differences in $A-G$ is as follows:

cycle	1	-1	2	-2	3	-3	other arcs
A	1		(m-1)/2	(m-3)/2		1	
B	3	1	(m-5)/2	(m-3)/2			
C	m-4		3		1		
D		m-3		1		2	
E					m-1		(7,25)
F						m-1	(m,m-18)
G	3	4	5	5	4	3	(m-18,P,m), (25,7)

In Case 4, distribution of differences in $A-G$ is as follows:

cycle	1	-1	2	-2	3	-3	other arcs
A			3		(2m-5)/3	(m-4)/3	
B		2	1		(m-7)/3	(2m-2)/3	
C	m-5		5				
D		m-4		3		1	
E			(m-9)/2	(m+3)/2	1	1	(10,22)
F			(m-3)/2	(m-9)/2	2	2	(m+4,2),(m-4,m-13)
G	10	7	2	5	6	3	(2,P,m-4),(22,m+4),(m-13,10)

Theorem 3 *The covering number $C(v,m)$ is determined for any fixed odd positive integer m and positive integer v, $m \le v \le m+6$.*

Theorem 4 *Let $m = p^r, pq, p^2q, p^3q$, where p and q are distinct odd primes. If there exist optimal (v,m)-DCC, for $v \in [m+7, 2m-4] \bigcup [2m+5, 3m-4]$, then there exist optimal (v,m)-DCC for $v \ge m$.*

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完全有向图的奇长圈覆盖

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摘 要: 设 DK_v 表示完全有向对称图， $C(v,m)$ 表示覆盖 DK_v 的 m 长有向圈的最小圈数 (称为覆盖数). 对任意正整数 m 和 v, 当 $m \le v \le m+6$ 时，覆盖数 $C(v,m)$ 被确定.