Journal of Mathematical Research & Exposition
Vol.22, No.3, 401–406, August, 2002

The Ordinary Bailey Lemma and Riordan Chain *

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Abstract: The present paper is concerned with Bailey lemma which has been proved to be useful in the studies of hypergeometric function and Ramannujan-Rogers identities, etc. We will show that the Bailey lemma in ordinary form is in fact a Riordan chain of a particular Riordan group.

Key words: Riordan group/chain; Bailey lemma; q-hypergeometric series.

Classification: AMS(2000) 05A15/CLC O157.1

Document code: A **Article ID:** 1000-341X(2002)03-0401-06

The following result is called in the literature as Bailey lemma which plays a very important role in the studies of hypergeometric function, hypergeometric series, q-hypergeometric series, and Ramannujan-Rogers identities.

Theorem 1 (Bailey lemma [5]) *Let $\alpha=(\alpha_0,\alpha_1,\alpha_2,\cdots,\alpha_n,\cdots)$ and $\beta=(\beta_0,\beta_1,\beta_2,\cdots,\beta_n,\cdots$ be two arbitrary infinite complex sequences satisfying*

$$\beta_n=\sum_{k=0}^{n}\frac{1}{(q;q)_{n-k}(aq;q)_{n+k}}\alpha_k. \tag{1}$$

Then there exists a pair of new sequences $\alpha'=(\alpha_0',\alpha_1',\alpha_2',\cdots,\alpha_n',\cdots)$ and $\beta'=(\beta_0',\beta_1',\beta_2',\cdots,\beta_n',\cdots)$ such that

$$\beta_n'=\sum_{k=0}^{n}\frac{1}{(q;\ q)_{n-k}(aq;\ q)_{n+k}}\alpha_k',$$

where $\alpha_k'=\frac{(\rho_1;\ q)_k(\rho_2;\ q)_k}{(aq/\rho_1;q)_k(aq/\rho_2;\ q)_k}(\frac{aq}{\rho_1\rho_2})^k\alpha_k$, $\beta_n'=\sum_{k\ge 0}\frac{(\rho_1;\ q)_k(\rho_2;\ q)_k(aq/\rho_1\rho_2;\ q)_{n-k}}{(q;q)_{n-k}(aq/\rho_1;\ q)_n(aq/\rho_2;\ q)_n}\beta_k$.

Throughout this paper, $(a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k)$, the vector $\alpha=(\alpha_0,\alpha_1,\alpha_2,\cdots,\alpha_n,\cdots)$ denotes an infinite complex sequence consisting of α_n. As known to us, it is D.Stinson who put forward firstly the idea of Bailey chain based on Bailey lemma. G.Andrews used Bailey lemma to investigate Rammanujan-type identity as well as q-series transformation

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and summation [5]. We have introduced the Riordan chain of a given Riordan group in the paper [1,2]. Comparing the definition of Riordan chain of Riordan group with Bailey lemma, one cannot help asking the question: does there exist any relationship between them ? This is just the theme of the present paper. For this, we assume the reader are familiar to Riordan group without going into details for them, and only to restate the definition of Riordan chain in here to make this paper self-contained.

Definition 1 *Let M_R be the Riordan group and $A=(a_{n,k})=(g(x),\ f(x))$, $B=(b_{n,k})=(c_1(t),\ d_1(t))$, and $C=(c_{n,\ k})=(c_2(t),\ d_2(t))\in M_R$. Suppose that $\alpha=(\alpha_0,\alpha_1,\alpha_2,\cdots,\alpha_n,\cdots)$ and $\beta=(\beta_0,\beta_1,\beta_2,\cdots,\beta_n,\cdots)$ be two arbitrary complex sequences satisfying $\beta_n=\sum_{k=0}^{n}a_{n,k}\ \alpha_k$. To construct two new sequences $\alpha'=(\alpha'_0,\alpha'_1,\alpha'_2,\cdots,\alpha'_n,\cdots)$ and $\beta'=(\beta'_0,\beta'_1,\beta'_2,\cdots,\beta'_n,\cdots)$ such that*

$$\begin{cases} \alpha'_n=\sum_{k=0}^{n}b_{n,\ k}\alpha_k, \\ \beta'_n=\sum_{k=0}^{n}c_{n,k}\beta_k. \end{cases}$$

If $\alpha'=(\alpha'_0,\alpha'_1,\alpha'_2,\cdots,\alpha'_n,\cdots)$ and $\beta'=(\beta'_0,\beta'_1,\beta'_2,\cdots,\beta'_n,\cdots)$ still satisfies the relation $\beta'_n=\sum_{k=0}^{n}a_{n,k}\ \alpha'_k$. Then we call such a process to be a Riordan chain, denoted by $(A;\ B,C)$. This definition can be displayed by the following commutative digraph, where $\alpha\overset{A}{\mapsto}\beta$ means that $A\alpha=\beta$.

$$\begin{array}{ccc} \alpha & \xrightarrow{B} & \alpha' \\ A\downarrow & & \downarrow A \\ \beta & \overrightarrow{C} & \beta' \end{array}$$

The following fundamental result established in [1] is an important characterization of Riordan chain.

Theorem 2[1] *Let M_R be the Riordan group and $A=(g(x),\ f(x))$, $B=(c_1(t),\ d_1(t))$, and $C=(c_2(t),\ d_2(t))\in M_R$. Then there exists a Riordan chain $(\mathcal{A};\mathcal{B},\mathcal{C})$ if and only if*

$$\begin{cases} c_1(f(t))\ g(t)=c_2(t)\ g(d_2(t)), \\ d_1(f(t))=f(d_2(t)). \end{cases} \tag{2}$$

Now, we might see clearly that Bailey lemma is in fact the Riordan chain of Riordan group, this fact is represented by Theorems 3 (while $a\neq 1$) and 5 ($a=1$), respectively. Observe that if we replace β_n by $\beta_n/n!(1-q)^n(1-a)^n$ and α_n by $(1-q)^n(1-a)^n\alpha_n/n!$ and let $q\longrightarrow 1-$. Then (11) in Bailey lemma turns out to be the ordinary form (with respect to q-series form) of Bailey lemma, rephrased as below

$$\beta_n=\sum_{k=0}^{n}\binom{n}{k}\alpha_k.$$

Furthermore, by using Theorem 2 we can set up a Riordan chain,i.e., an ordinary form of Bailey lemma.

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Theorem 3 *Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ and $\beta = (\beta_0, \beta_1, \beta_2, \cdots, \beta_n, \cdots)$ be two arbitrary complex sequences satisfying $\beta_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k$. Then there exists a pair of new sequences $\alpha' = (\alpha'_0, \alpha'_1, \alpha'_2, \cdots, \alpha'_n, \cdots)$ and $\beta' = (\beta'_0, \beta'_1, \beta'_2, \cdots, \beta'_n, \cdots)$ such that*

$$\beta'_n = \sum_{k=0}^{n} \binom{n}{k} \alpha'_k,$$

where $\alpha'_k = \frac{(1-\rho_1)^k(1-\rho_2)^k}{(\rho_1-a)^k(\rho_2-a)^k} a^k\ \alpha_k$, $\beta'_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(1-\rho_1)^k(1-\rho_2)^k(\rho_1\rho_2-a)^{n-k}(1-a)^{n-k}}{(\rho_1-a)^n(\rho_2-a)^n} \beta_k$.

We have discussed in [1] the problem of Riordan chain of binomial coefficient $\binom{n}{k}$. Now, we might uniform the main result in [1] with Theorem 3 and obtain a more general version.

Theorem 4 $\forall A = (\binom{n}{k}) = (\frac{1}{1-t}, \frac{t}{1-t})$, $B = (a^k \delta_{n,k}) = (1, at)$, and $C = (c_2(t), d_2(t)) \in M_R$. *Then there exists the Riordan chain $(A; B, C)$ if and only if*

$$c_2(t) = \frac{1}{1-(1-a)t} \text{ and } d_2(t) = \frac{a\ t}{1-(1-a)t}.$$

Proof Setting all conditions into (2), one can obtain directly that

$$\begin{cases} \frac{c_2(t)}{1-d_2(t)} = \frac{1}{1-t}, \\ \frac{d_2(t)}{1-d_2(t)} = \frac{a\ t}{1-t}. \end{cases}$$

Solving this system of equations leads to the desired results directly.

Corollary 1 (Bailey lemma in ordinary form) *Let M_R be the Riordan group and $A = (a_{n,k}) = (g(x), f(x))$, $B = (b_{n,k}) = (c_1(t), d_1(t))$, and $C = (c_{n,k}) = (c_2(t), d_2(t))$. Suppose that $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ and $\beta = (\beta_0, \beta_1, \beta_2, \cdots, \beta_n, \cdots)$ be two arbitrary complex sequences satisfying $\beta_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k$. Suppose that two new sequences $\alpha' = (\alpha'_0, \alpha'_1, \alpha'_2, \cdots, \alpha'_n, \cdots)$ and $\beta' = (\beta'_0, \beta'_1, \beta'_2, \cdots, \beta'_n, \cdots)$ be given by*

$$\begin{cases} \alpha'_k = a^k \alpha_k, \\ \beta'_n = \sum_{k=0}^{n} \binom{n}{k} a^k (1-a)^{n-k} \beta_k. \end{cases}$$

Then α' and β' still satisfies the relation $\beta'_n = \sum_{k=0}^{n} \binom{n}{k} \alpha'_k$. The technique of generating function will make such the process of chain-structure displayed very clearly.

Corollary 2 *Let α and β be given as above. Suppose that $f(t) = \sum_{n \ge 0} \alpha_n / n! t^n$, $g(t) = \sum_{n \ge 0} \beta_n / n! t^n$, such that $g(t) = e^t f(t)$. Further, let*

$$\begin{cases} f_1(t) = f(at), \\ g_1(t) = e^{(1-a)t} g(at). \end{cases}$$

Then it still holds that $g_1(t) = e^t f_1(t)$.

It is worth of pointing that this structure remains unchanged after arbitrary m times iteration.

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Corollary 3 (m-th iteration of Bailey lemma) *Let M_R be the Riordan group and $A = (a_{n,k}) = (g(x), f(x)), B = (b_{n,k}) = (c_1(t), d_1(t))$, and $C = (c_{n,k}) = (c_2(t), d_2(t))$. Suppose that $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ and $\beta = (\beta_0, \beta_1, \beta_2, \cdots, \beta_n, \cdots)$ be two arbitrary complex sequences satisfying $\beta_n = \sum_{k=0}^n \binom{n}{k}\alpha_k$. Define two new sequences $\alpha' = (\alpha'_0, \alpha'_1, \alpha'_2, \cdots, \alpha'_n, \cdots)$ and $\beta' = (\beta'_0, \beta'_1, \beta'_2, \cdots, \beta'_n, \cdots)$ by*

$$\begin{cases} \alpha'_k = a^{mk}\alpha_k, \\ \beta'_n = \sum_{k=0}^n \binom{n}{k} a^{mk}(1-a^m)^{n-k}\beta_k. \end{cases}$$

Then $\alpha' = (\alpha'_0, \alpha'_1, \alpha'_2, \cdots, \alpha'_n, \cdots)$ and $\beta' = (\beta'_0, \beta'_1, \beta'_2, \cdots, \beta'_n, \cdots)$ still satisfies the relation $\beta'_n = \sum_{k=0}^n \binom{n}{k}\alpha'_k$.

Remark 1 In Theorem 4, if we chose $a = \frac{(1-\rho_1)(1-\rho_2)}{(\rho_1-b)(\rho_2-b)}b$. Then

$$\begin{aligned} 1-a &= 1 - \frac{(1-\rho_1)(1-\rho_2)}{(\rho_1-b)(\rho_2-b)}b = \frac{(\rho_1-b)(\rho_2-b) - b + b\rho_1 + b\rho_2 - b\rho_1\rho_2}{(\rho_1-b)(\rho_2-b)} \\ &= \frac{\rho_1\rho_2 + b^2 - b - b\ \rho_1\rho_2}{(\rho_1-b)(\rho_2-b)} = \frac{(\rho_1\rho_2-b)(1-b)}{(\rho_1-b)(\rho_2-b)}. \end{aligned}$$

This conclusion leads directly to Theorem 3. More generally, from Andrews' result we know that 2-th iteration of Bailey lemma yields Watson's transformation between ${}_8\Phi_7$ and ${}_4\Phi_3$. Now we can give its similar result for $m = 2$ in Corollary 3.

Corollary 4 *Let $\alpha = (\alpha_0,\ \alpha_1,\ \alpha_2, \cdots,\ \alpha_n, \cdots)$ and $\beta = (\beta_0,\ \beta_1,\ \beta_2, \cdots, \beta_n, \cdots)$ be two arbitrary complex sequences satisfying $\beta_n = \sum_{k=0}^n \binom{n}{k}\alpha_k$. Then there exists a pair of new sequences $\alpha' = (\alpha_0',\ \alpha_1',\ \alpha_2', \cdots,\ \alpha_n', \cdots)$ and $\beta' = (\beta_0',\ \beta_1',\ \beta_2', \cdots,\ \beta_n', \cdots)$ such that*

$$\beta'_n = \sum_{k=0}^n \binom{n}{k}\alpha'_k,$$

where $\alpha'_k = \{\frac{(1-\rho_1)(1-\rho_2)}{(\rho_1-b)(\rho_2-b)}b\}^2\alpha_k$, $\beta'_n = \sum_{k=0}^n \binom{n}{k}(1-b^2)^{n-k}\ (\rho_1\rho_2-b)^{n-k}(1-\rho_1)^{2k}(1-\rho_2)^{2k} \times \frac{\{(\rho_1-\frac{2b}{1+b})(\rho_2-\frac{2b}{1+b})+b\frac{(1-b)^2}{(1+b)^2}\}^{n-k}}{(b-\rho_1)^{2n}(b-\rho_2)^{2n}}b^{2k}\beta_k$.
Note: for $m \geq 3$ large enough, such result will become more and more complicated.

Now, we consider the left case that $a = 1$. Evidently, the preceding argument is no longer useful for this case. However, with the help of Riordan group, we are still able to set up the corresponding result.

Theorem 5 *Let $\alpha = (\alpha_0,\ \alpha_1,\ \alpha_2, \cdots,\ \alpha_n, \cdots)$ and $\beta = (\beta_0,\ \beta_1,\ \beta_2, \cdots, \beta_n, \cdots)$ be two arbitrary complex sequences satisfying $\beta_n = \sum_{k=0}^n \binom{n}{k}\alpha_k$. Then there exists a pair of new sequences α' and β' such that*

$$\beta'_n = \sum_{k=0}^n \binom{n}{k}\alpha'_k,$$

where $\alpha'_k = a^k\ \alpha_k, \beta'_n = \sum_{k=0}^n \sum_{i=k}^n (-1)^{i-k}a^i((1-a)n + (1+a)i + 1)\binom{n}{k}\binom{n}{k}\beta_k$.

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Proof Consider that

$$\binom{n}{k} = [t^n]\frac{1}{\sqrt{1-4t}}(\frac{1-2t-\sqrt{1-4t}}{2t})^k.$$

Then in Theorem 2, if one chooses $A = (\frac{1}{\sqrt{1-4t}}, (\frac{1-2t-\sqrt{1-4t}}{2t})) := (g(t), f(t))$ and $B := (c_1(t), d_1(t)) = (1, at)$ where a is an arbitrary constant. As the argument states, it only needs to find $C = (c_2(t), d_2(t))$, that is, to solve the system of equations (2). At first, let $\sqrt{1-4t} = e$. Then from (2) it yields a relation between $d_2(t)$ and e: $d_2(t) = (1-e^2)/4$. Setting it into (2) again, we work out $d_2(t)$ firstly and then $c_2(t)$:

$$\begin{cases} c_2(t) = g(t)\frac{1-af(t)}{1+af(t)}, \\ d_2(t) = \frac{af(t)}{(1+af(t))^2}. \end{cases}$$

This implies the desired result.

Theorem 5 can be stated equivalently in terms of Riordan group.

Corollary 5 *$\forall A = (g(t), f(t)) \in M_R$ and a being constant, there always exists a Riordan chain*

$$(A; B, C) = ((\frac{1}{\sqrt{1-4t}}, \frac{1-2t-\sqrt{1-4t}}{2t}); (1, at), (g(t)\frac{1-af(t)}{1+af(t)}, \frac{af(t)}{(1+af(t))^2}))$$

in M_R.

Remark 2 Actually, the Riordan chain is equivalent to the fact that there exist three matrices A, B, and $C \in M_R$ such that $AB = CA$. Thus, to study the Riordan chain is to find all matrixes A, B and C satisfying $AB = CA$. Here are two propositions satisfied by the entries of A, B and C.

Theorem 6 *Suppose that $A = (a_{n,\, k})$, $B = (b_{n,\, k})$, and $C = (c_{n,\, k}) \in M_R$ such that $AB = CA$. Then $b_{n,n} = c_{n,n}$ for all $n \geq 0$.*

Theorem 7 *Suppose that $A = (a_{n,\, k})$, $B = (b_{n,\, k})$, and $C = (c_{n,\, k}) \in M_R$ such that $AB = CA, b_{n,\, k} = 0$ for all $n \neq k$. Then $c_{n,\, k} = \sum_{j=k}^{n} a_{n,j}\, b_{j,j}\, a^{-1}_{j,\, k}$ for all $n,\ k \geq 0$.*

Proof Since that $AB = CA$ is equivalent to $C = ABA^{-1}$, we have that

$$c_{n,k} = \sum_{j=k}^{n}\{\sum_{i=j}^{n} a_{n,\, i} b_{i,j}\} a^{-1}_{j,\, k} = \sum_{j=k}^{n} a_{n,j} b_{j,j}\ a^{-1}_{j,k},$$

where $A^{-1} = (a^{-1}_{j,k})$.

This conclusion covers a lot. To justify this claim, let us see the following example.

Example (Riordan chain)

$$a_{n,k} = \binom{n}{k},\ a^{-1}_{n,\, k} = (-1)^{n-k}\binom{n}{k},$$

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and $b_{n,k} = a^k \delta_{n,k}$. Direct calculation yields to $c_{n,k} = \binom{n}{k} a^k (1-a)^{n-k}$. Thus, the relation $AB = CA$ equals that

$$\binom{n}{k} a^k (1-a)^{n-k} = \sum_{j=k}^{n} \binom{n}{j} a^j (-1)^{j-k} \binom{j}{k},$$

which is a simple but fundamental relation in combinatorial analysis, which serves as a witnesses to what we mentioned above is correct. Also it tells that Bailey lemma belongs to the category of Riordan chain.

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普通型 Bailey 引理和 Riordan 链

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摘 要: 本文主要研究在超几何函数和 Ramanujan-Rogers 类型恒等式有非常重要作用的. Beilay 引理和 Riordan 链的关系，证明了普通型 Bailey 引理本质上是一个特殊 Riordan 群的 Riordan 链.

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