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Sensitivity Analysis in Vector Optimization under Benson Proper Efficiency *

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Abstract: The behavior of the perturbation map is analyzed quantitatively by using the concept of contingent derivatives for set-valued maps under Benson proper efficiency. Let $W(u) = P\min[G(u), S], y^\wedge \in W(u^\wedge)$. It is shown that, under some conditions, $DW(u^\wedge, y^\wedge) \subset P\min[DG(u^\wedge, y^\wedge), S]$, and under some other conditions, $DW(u^\wedge, y^\wedge) \supset P\min[DG(u^\wedge, y^\wedge), S]$.

Key words: sensitivity analysis; perturbation maps; contingent derivatives; Benson proper efficiency.

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1. Introduction

It is well known that stability and sensitivity analysis is not only theoretically interesting but also practically important in optimization theory. A number of useful results have been obtained in usual scalar optimization. See, for example, B.Bank, J.Guddat, D.Klatte, B.Kumer and K.Tammer[1], and R.T.Rockafellar[2]. Here, stability means the qualitative analysis of the perturbation (or marginal) function (or map) of a family of parametrized optimization problems, and sensitivity means the quantative analysis, that is, the study of derivatives of the perturbation function.

For vector optimization ,the optimal values are not unique, and hence we should consider a set-valued perturbation map. Though several concepts of derivative of set-valued map were proposed[3],[5], the concept of contingent derivative is the most adequate for our purposes[5]. Along with this thought, many developments[6-9] are obtained. On the other hand, we note that though the discussions are delt with the sensitivity of a given efficient element, the given element is required to be a properly efficient element of the perturbation map (or function),which implies that the proper efficiency is an essential property

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and we should discuss the stability and sensitivity for a given proper efficient element, for example, Benson proper efficient element. The stability problems about Benson proper efficiency have been investigated by Wei L.Y.,Huang Z.J. and Mei J.L. in [10]. The purpose of this paper is to discuss the sensitivity for the Benson proper efficient element of a perturbation map.

2. Contingent derivative and the Benson proper efficient elements of set-valued map

Throughout this section, U and Y are two Banach spaces, F is a set-valued map from U to $Y, S \subset Y$ is a closed, convex and pointed cone with interior $\mathrm{int}S \neq \emptyset$.

By θ_Y we denote the zero element of linear vector spaces Y and by $\mathrm{cl}A$ we denote the closure of A.

Let $A \subset Y$ be a nonempty subset. The generated cone of A is defined as

$$\mathrm{cone}(A) = \{\alpha a : \alpha \geq 0, a \in A\}.$$

It is well known that $\mathrm{cone}(A)$ is a nonempty cone. It is also a convex cone if A is a convex set.

A set-valued map $F : U \to 2^Y$ is called S-convex on U, if, for any $x_1, x_2 \in U$, and for any $\lambda \in [0,1]$, we have $\lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + S$.

For a set-valued map $F : U \to 2^Y$, we define the epigraph of F by

$$\mathrm{epi}F = \{(u,y) : u \in U, y \in F(u) + S\}.$$

A well known result is that F is S-convex if and only if $\mathrm{epi}F$ is a convex set.

Definition 1[3] *Let A be a nonempty subset of Banach space $U, u^\wedge \in \mathrm{cl}A$. The set $T_A(u^\wedge) \subset U$ defined by $T_A(u^\wedge) = \bigcap_{\varepsilon>0} \bigcap_{\alpha>0} \bigcup_{0<h\leq\alpha} (\frac{1}{h}(A-u^\wedge)+\varepsilon B)$ is called the contingent cone of A at $u^\wedge$. Where B is the unit ball in U. In other words, $u \in T_A(u^\wedge)$ if and only if there exist $h_n > 0, h_n \to +\infty (n \to +\infty), u_n \in A, u_n \to u^\wedge (n \to +\infty)$,such that $u = \lim_{n\to+\infty} h_n(u_n - u^\wedge)$.*

It is well known that $T_A(u^\wedge)$ is a closed convex cone when A is a convex set.

Definition 2 *Let $y_0 \in F(u_0)$. We call a set-valued map $DF(u_0,y_0) : U \to 2^Y$ the contingent derivative of F at (u_0,y_0) if $\mathrm{epi}DF(u_0,y_0) = T_{\mathrm{epi}F}(u_0,y_0)$.*

Definition 3 *Let $A \subset Y$ be a nonempty set, we define*

$$P\min[A,S] = \{y \in A : (-S) \cap \mathrm{clcone}(A+S-y) = \{\theta_Y\}\},$$

$$P\max[A,S] = \{y \in A : S \cap \mathrm{clcone}(A-S-y) = \{\theta_Y\}\},$$

and

$$\min[A,S] = \{y \in A : A \cap (y-S) = \{y\}\},$$

where $\mathrm{cl}A$ is the closure of A. It is not difficult to show that

$$P\min[A,S] \subset \min[A,S]$$

is always true.

We call $P\min[A,S]$ the Benson proper efficient set of A about pointed cone S. When $y^\wedge \in P\min[A,S]$, we call $y^\wedge$ a Benson proper effieient point of set A.

Definition 4[12] *A base for $A \subset Y$ is a nonempty convex subset Q of A with $\theta_Y \notin Q$ such that every $a \in A, a \neq \theta_Y$, has a unique representation of the form αb,where $b \in Q$ and $\alpha > 0$.*

A set-valued map $F : U \to 2^Y$ is said to be lower semicontinuous at $u^\wedge \in U$,if $u_k \to u^\wedge$ and $y^\wedge \in F(u^\wedge)$ imply the existence of a integer K and a sequence $\{y^k\} \subset Y$ such that $y^k \in F(u^k)$ for $k \geq K$ and $y^k \to y^\wedge (k \to +\infty)$.

Definition 5[6] *Let A be a nonempty set in U, U^* denote the dual of U. The normal cone $N_A(u^\wedge)$ to A at $u^\wedge$ is the negative polar cone of the tangent cone $T_A(u^\wedge)$,i.e.,*

$$N_A(u^\wedge) = [T_A(u^\wedge)]^0 = \{\varphi \in U^* : \varphi(u) \leq 0, u \in T_A(u^\wedge)\}.$$

When A is a convex set and $u^\wedge \in A$ we have $N_A(u^\wedge) = \{\varphi \in U^* : \varphi(u^\wedge) \geq \varphi(u), u \in A\}$.

Definition 6[6] *Let $A+S$ be a nonempty subset in Y. If $y^\wedge \in P\min[A,S]$ satisfies*

$$N_{A+S}(y^\wedge) \subset \mathrm{int} S^0 \bigcup \{\theta_Y\},$$

then $y^\wedge$ is called the normally Benson proper efficient point of A.

3. Contingent derivative of the perturbation map

Let $G(u)$ be a set-valued map from U to Y with U the perturbation parameter vector space and Y the objective space. We define another set-valued map W from U to Y by

$$W(u) = P\min[G(u), S] \tag{1}$$

for every $u \in U$, and call it the perturbation map, since it is a generalization of the perturbation map in scalar optimization, vector optimization, and set-valued optimization.

We call $(u^\wedge, y^\wedge)$ a Benson proper efficient element of (1) if $y^\wedge \in P\min[G(u^\wedge), S]$.

The purpose of this section is to investigate the relations between the contingent derivatives of W and that of G.

Lemma 1[6] *If $F(u) : U \to 2^Y$ is an S-convex set-valued map and $u^\wedge \in \mathrm{int} U$, then $F+S$ is lower semicontinuous at $u^\wedge$.*

Lemma 2[13] *For a cone $Q \subset Y$ and its dual cone $Q^* = \{\varphi \in Y^* | \varphi(q) \geq 0, q \in Q\}$, we have $\varphi(q) > 0$ for $\varphi \in Q^* \backslash \{\theta_{Y^*}\}, q \in \mathrm{int} Q$, and $\varphi \in \mathrm{int} Q^*, q \in Q \backslash \{\theta_Y\}$.*

Lemma 3 $G(u) \subset W(u) + S$.

Proof Suppose to the contrary, there exists $y \in G(u)$ such that $y \notin W(u) + S$, then for any $s \in S$ we have $y - s \notin W(u) + S$, which implies $y - s \notin P\min(G(u), S)$ and

$$\mathrm{clcone}(G(u) + S - y + s) \bigcap (-S \backslash \{\theta_Y\}) \neq \emptyset.$$

Let $\alpha \in \text{clcone}(G(u)+S-y+s)\bigcap(-S\backslash\{\theta_Y\})$. Then there exist $t_n \geq 0, y_n \in G(u), s_n \in S$, such that for any $s \in S$, $\alpha = \lim\limits_{n\to+\infty} t_n(y_n+s_n-y+s) \in -S\backslash\{\theta_Y\}$.

Let $\varphi \in \text{int}S^*$. Then by Lemma 2 we know

$$0 > \varphi(\alpha) = \lim_{n\to+\infty} t_n(\varphi(y_n)+\varphi(s_n)-\varphi(y)+\varphi(s))$$

holds for any $s \in S$. Hence there exists $N > 0$ such that when $n \geq N$,

$$t_n(\varphi(y_n)+\varphi(s_n)-\varphi(y)+\varphi(s)) < 0$$

holds for any $s \in S$. In particular, we have for any $s \in S$,

$$\varphi(y_N)+\varphi(s_N)-\varphi(y)+\varphi(s)) < 0,$$

which is not correct since S is a cone. Hence Lemma 3 holds.

Theorem 1 *Let $G(u)$ be an S-convex set-valued map, $(u^\wedge, y^\wedge)$ be a Benson proper element of* (1). *Then*

$$DG(u^\wedge, y^\wedge)(u) \subset DW(u^\wedge, y^\wedge)(u) \tag{2}$$

holds for any $u \in U$. Hence $P\min[DG(u^\wedge, y^\wedge)(u), S] \subset DW(u^\wedge, y^\wedge)(u)$.

Proof Let $y \in DG(u^\wedge, y^\wedge)(u)$. Then by Definition 2 ,there exist sequence $\{u_n\} \subset U, u_n \to u^\wedge, \{y_n\} \subset Y, y_n \in G(u_n)+S, y_n \to y^\wedge, t_n > 0, t_n \to +\infty (n \to +\infty)$,such that

$$(u, y) = \lim_{n\to+\infty} t_n((u_n, y_n)-(u^\wedge, y^\wedge)).$$

By Lemma 3 we know that $G(u) \subset W(u)+S$. Hence $G(u_n)+S \subset W(u_n)+S$, and $y_n \in W(u_n)+S$. Therefore $y \in DW(u^\wedge, y^\wedge)(u)$, and (2) holds.

Theorem 2 *Let $u^\wedge \in \text{int}U, y^\wedge$ be a normally Benson proper efficient point of $DG(u^\wedge, y^\wedge)(u)$, S have a compact base, U^* and Y^* are * weak compact. Then*

$$DW(u^\wedge, y^\wedge)(u) \subset P\min[DG(u^\wedge, y^\wedge)(u), S], u \in U.$$

Proof Let $y \in DW(u^\wedge, y^\wedge)(u)$. Then $y \in DG(u^\wedge, y^\wedge)(u)$.

If $y \notin P\min[DG(u^\wedge, y^\wedge)(u), S]$, then $\text{clcone}(DG(u^\wedge, y^\wedge)(u)+S-y)\bigcap(-S\backslash\{\theta_Y\}) \neq \emptyset$.

Let $\alpha \in \text{clcone}(DG(u^\wedge, y^\wedge)(u)+S-y)\bigcap(-S\backslash\{\theta_Y\})$. Then there exist $y_n \in DG(u^\wedge, y^\wedge)(u)$, $s_n \in S, t_n > 0$, such that $\alpha = \lim\limits_{n\to+\infty} t_n(y_n+s_n-y) \in -S\backslash\{\theta_Y\}$.

By Lemma 2 we know for any $\psi \in N_{DG(u^\wedge, y^\wedge)+S}(y^\wedge)$

$$\psi(\alpha) = \lim_{n\to+\infty} t_n(\psi(y_n)+\psi(s_n)-\psi(y)) > 0.$$

Hence, there exists $\overline{y} \in DG(u^\wedge, y^\wedge)(u)$ such that

$$\psi(\overline{y}-y) > 0. \tag{3}$$

Since $\overline{y} \in DG(u^\wedge, y^\wedge)(u)$, there exist $\{\overline{h}_k\} \in \text{int}R_+, \overline{y}_k \in Y, \overline{u}_k \in U (k = 1, 2, \cdots)$, such that $\overline{u_k} \to u, \overline{y}_k \to \overline{y}$, and for each k, $y^\wedge + \overline{h}_k\overline{y}_k \in G(u^\wedge + \overline{h}_k\overline{u}_k)+S$.

On the other hand, by $y \in DW(u^\wedge, y^\wedge)(u)$ there exist $h_k > 0(k = 1, 2, \cdots), u_k \in U, y_k \in Y(k = 1, 2, \cdots)$, such that $h_k \to 0, u_k \to u, y_k \to y \quad (k \to \infty)$, and for any k

$$y^\wedge + h_k y_k \in W(u^\wedge + h_k u_k) + S.$$

Since $h_k \to 0$, we may assume $h_k \leq \overline{h}_k$ by taking a subsequence if necessary. Since $y^\wedge + h_k y_k \in W(u^\wedge + h_k u_k) + S$, there exist $s_k \in S(k = 1, 2, \cdots)$ such that

$$y^\wedge + h_k y_k \in W(u^\wedge + h_k u_k) + s_k.$$

Let $s_k = a_k q_k$. Then $y^\wedge + h_k y_k - a_k q_k \in W(u^\wedge + h_k u_k)$, which implies that $(u^\wedge + h_k u_k, y^\wedge + h_k y_k - a_k q_k)(k = 1, 2, \cdots)$ is boundary points of the convex set epiG. Hence, there exist $(\varphi_k, \psi_k) \in U^* \times Y^*, (\varphi_k, \psi_k) \neq \theta_{U^* \times Y^*}(k = 1, 2, \cdots)$, such that

$$\varphi_k(u^\wedge + h_k u_k) + \psi_k(y^\wedge + h_k y_k - a_k q_k) \geq \varphi_k(u') + \psi_k(y'), (u', y') \in \text{epi}G. \tag{4}$$

By the assumption that U^* and Y^* are * weak compact, without loss of generality, we assume that $(\varphi_k, \psi_k) \to^{*w} (\varphi, \psi) \neq \theta_{U^* \times Y^*}, (k \to +\infty)$. By taking limits of (4) with $k \to +\infty$, we have

$$\varphi(u^\wedge) + \psi(y^\wedge - aq) \geq \varphi(u') + \psi(y'), (u', y') \in \text{epi}G, \tag{5}$$

where $a = \lim_k a_k, q = \lim_k q_k$. If $\psi = \theta_{Y^*}$, then $\varphi \neq \theta_{U^*}$ and $\varphi(u^\wedge) \geq \varphi(u'), u' \in U$.Since $u^\wedge \in \text{int}U$, then we obtain by $\varphi(u^\wedge) \geq \varphi(u')$ that $\varphi = 0$. Therefore $\psi \neq \theta_{Y^*}$.

By Lemma 1 $G(u)$ is lower semicontinuous at $u^\wedge$. For any $y^\sim \in G(u^\wedge) + S$ there exist a sequence $\{y_k^\sim\} \subset Y$ such that $y_k^\sim \to y^\sim$ and an integer $K > 0$ such that

$$y_k^\sim \in G(u^\wedge + h_k u_k) + S, k \geq K. \tag{6}$$

By (4) we know for $k \geq K$,

$$\varphi_k(u^\wedge + h_k u_k) + \psi_k(y^\wedge + h_k y_k - a_k q_k) \geq \varphi_k(u^\wedge + h_k u_k) + \psi_k(y_k^\sim).$$

By taking $k \to \infty$ in above formula we have

$$\psi(y^\wedge - aq) \geq \psi(y^\sim)$$

and

$$\psi \in N_{G(u^\wedge)+S+aq}(y^\wedge) \subset N_{G(u^\wedge)+S}(y^\wedge).$$

Note that $y^\wedge + \overline{h}_k \overline{y}_k \in G(u^\wedge + \overline{h}_k \overline{u}_k) + S, y^\wedge \in G(u^\wedge)$, and $h_k \leq \overline{h}_k$, we have

$$y^\wedge + h_k \overline{y}_k \in G(u^\wedge + h_k \overline{u}_k) + S.$$

By the S-convexity of $G(u)$ and (4) we have

$$\varphi_k(u^\wedge + h_k u_k) + \psi_k(y^\wedge + h_k y_k - a_k q_k) \geq \varphi_k(u^\wedge + h_k \overline{u}_k) + \psi_k(y^\wedge + h_k \overline{y}_k).$$

Let $k \to +\infty$. Then $\varphi(u)+\psi(y)-\psi(aq) \geq \varphi(u)+\psi(\overline{y})$, or $\psi(y)-\psi(aq) \geq \psi(\overline{y})$, which is a contradiction to (3).Therefore

$$y \in P\min[DG(u^\wedge, y^\wedge)(u), S].$$

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Benson 真有效意义下向量优化的灵敏度分析

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摘 要: 本文用关于集值映射的 Contingent 切导数定量地讨论了参数映射 $G(u)$ 在 Benson 真有效意义下的扰动情况. 记 $W(u) = P\min[G(u), S], y^\wedge \in W(u^\wedge)$, 则在某些条件下 $DW(u^\wedge, y^\wedge)(u) \subset P\min[DG(u^\wedge, y^\wedge)(u)]$, 而在另外一些条件下 $DW(u^\wedge, y^\wedge)(u) \supset P\min[DG(u^\wedge, y^\wedge)(u)]$.