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Generalized Vectorial Quasi-Equilibrium Problem on W-Space *

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Abstract: Generalized vectorial quasi-equilibrium problem and generalized set-valued quasi-equilibrium problem are introduced and the existence theorems of generalized vectorial quasi-equilibrium problem on W-space are obtained. As corollaries, existence theorems of four kinds of quasi-equilibrium problem are obtained, and these results generalize and improve correspond results in [1-7] .

Key words: generalized vectorial quasi-equilibrium; W-Space; set-valued map.

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1. Introduction and Preliminaries

Given a nonempty set X and bifunction $f : X \times X \to [-\infty, +\infty]$ and a set-valued map $M : X \to 2^X$, the scalar quasi-equilibrium problem(see [1-3]) is as follows:

$$\text{Find } x^* \in X,$$

such that $x^* \in M(x^*)$ and $f(x^*, y) \geq 0, \forall y \in M(x^*)$. When replacing the function$f$ by vectorial-valued map $\hat{f} : X \times X \to Z$ and replacing the range space R by a real topological vector space Z, $P \subset Z$ being a pointed closed convex cone with int$P \neq \emptyset$, the vector quasi-equilibrium problem (see [4]) is as follows : Find $x^* \in X$, such that $x^* \in M(x^*)$ and $\hat{f}(x^*, y) \in P, \forall y \in M(x^*)$, or find $x^* \in X$, such that $x^* \in M(x^*)$ and $\hat{f}(x^*, y) \notin -\text{int}P, \forall y \in M(x^*)$. When replacing $\hat{f}$ by a multi-valued map $F : X \times X \to 2^Z$, the set-valued quasi-equilibrium problem is Find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } F(x^*, y) \not\subset -\text{int}P, \quad \forall y \in M(x^*), \tag{1.1}$$

or find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } F(x^*, y) \subset P, \quad \forall y \in M(x^*). \tag{1.2}$$

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Let $C \subset Z$ be a nonempty cone, we will get the following set-valued quasi-equilibrium problem is Find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } F(x^*,y) \cap C \neq \emptyset, \quad \forall y \in M(x^*), \tag{1.3}$$

or find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } F(x^*,y) \subset C, \quad \forall y \in M(x^*). \tag{1.4}$$

With $C = Z \backslash -\mathrm{int}P$, (1.3) contains (1.1) , and with $C = P$, (1.4) contains (1.2) . Let $F^-(C) = \{(x,y) \in X \times X : F(x,y) \cap C \neq \emptyset\}, F^+(C) = \{(x,y) \in X \times X : F(x,y) \subset C\}$, both problems (1.3) and (1.4) can be written as Find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } (x^*,y) \in F^{-1}(C), \quad \forall y \in M(x^*), \tag{1.5}$$

with $F^{-1} = F^-$ for (1.3) and $F^{-1} = F^+$ for (1.4).

A set-valued map $C : X \to 2^Z$ is called to be a family of domination structures in Z, if for every $x \in X$, $C(x)$ is a nonempty cone in Z. Replacing C in (1.3) and (1.4) by a family of domination structure $C(x)$, we will obtain a more generalized set-valued equilibrium problem is Find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } F(x^*,y) \cap C(x^*) \neq \emptyset, \quad \forall y \in M(x^*). \tag{1.6}$$

or Find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } F(x^*,y) \subset C(x^*), \quad \forall y \in M(x^*). \tag{1.7}$$

Clearly (1.6) contains (1.3) and (1.7) contains (1.4). We let

$$F_x^-(C) = \{(x,y) \in X \times X : F(x,y) \cap C(x) \neq \emptyset\},$$

$$F_x^+(C) = \{(x,y) \in X \times X : F(x,y) \subset C(x)\},$$

both problems (1.6) and (1.7) can be written as Find $x^* \in X$, such that

$$x^* \in M(x^*) \text{ and } (x^*,y) \in F_{x^*}^{-1}(C), \quad \forall y \in M(x^*) \tag{1.8}$$

with $F_x^{-1} = F_x^-$ for (1.6) and $F_x^{-1} = F_x^+$ for (1.7). It is clearly that (1.8) contains (1.5).

By using fixed point theorem on W-space, the existence theorem for generalized quasi-vectorial equilibrium problem (1.8) on W-space are obtained, as it's corollaries, existence theorems of four kinds of quasi-equilibrium problem on W-space are obtained, and these results generalize and improved the existence theorems for vectorial quasi-equilibrium problem in [1–4] and generalize and improved the existence theorems for equilibrium problem in [5–7].

Definition 1.1[8] *Let X be Hausdorff topological space and $\{C_K\}$ a family of nonempty connected subsets of X induced by finite subsets K of X such that $K \subset C_K$, then we call $(X, \{C_K\})$ be a W-space; let $(X, \{C_K\})$ be a W-space, the subset $D \subset X$ is called*

W-convex if for any finite subset K of D, $C_K \subset D$; D is said to be weakly W-convex if for any finite subsets K of D, $C_K \cap D$ is connected, i.e., $(D, \{C_{K\cap D} \cap D\})$ is a W-Space.

Definition 1.2[9] *Let $(X, \{C_K\})$ be a W-space, Y be a topological spaces, a set-valued map $F : X \to 2^Y$ is said to be W-KKM map, if for any $x_1, x_2 \in X$, $F(C_{\{x1,x2\}}) \subset F(x_1) \cup F(x_2)$, where $F(C_{\{x1,x2\}}) = \cup_{x\in C\{x1,x2\}} F(x)$.*

Lemma 1.1[11] *Let X, Y be two Hausdorff topological spaces, and $F : X \to 2^Y$ be a set-valued map. Then F is lower semi-continuous iff for any open subset $V \subset Y$, $F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ is a open set in X.*

Lemma 1.2 *Let $(X, \{C_K\})$ be a W-space, Y is a nonempty subset of X, and let the set-valued map $F : X \to 2^Y$ satisfy the following conditions: (i) the map F is nonempty open-valued and W-convex valued and F is a lower semi-continuous map; (ii) $\forall y \in X, X\backslash F^{-1}(y)$ is a W-convex set; (iii) If F is transfer closed-valued, and there exists $x_0 \in X$ such that $\mathrm{Cl}(F(x_0))$ is compact, then there exists $x^* \in Y$ such that $x^* \in F(x^*)$.*

Proof For any finite set $K \subset X$, $\cap_{x\in K} F(x)$ is connected from F is a W-convex valued, we can prove F is a W-KKM map, in fact, assume F isn't a W-KKM map , then there exists $x_1, x_2 \in X$, such that

$$F(C_{\{x1,x2\}}) \not\subset F(x_1) \cup F(x_2),$$

then there exists $y^* \in F(C_{\{x1,x2\}})$, such that $y^* \notin F(x_1)$ and $y^* \notin F(x_2)$, hence there exists $x^* \in C_{\{x1,x2\}}$, such that $y^* \in F(x^*)$, but $y^* \notin F(x_1)$ and $y^* \notin F(x_2)$, i.e.,

$$x_1, x_2 \in X\backslash F^{-1}(y^*), C_{\{x1,x2\}} \subset X\backslash F^{-1}(y^*)$$

from $X\backslash F^{-1}(y^*)$ is W-convex, hence $x^* \in X\backslash F^{-1}(y^*)$ from $x^* \in C_{\{x1,x2\}}$, i.e.,

$$y^* \notin F(x)^*),$$

which contradicts with $y^* \in F(x^*)$, hence F is a W-KKM map,. by Theorem 2 in [9], we know that $\{F(x) : x \in X\}$ has finite intersection property, by condition (iii) and from the proof of Lemma 4 of [7] , we have $\cap_{x\in X} F(x) \neq \emptyset$, thus there exists $x^* \in Y$, such that: $x^* \in F(x^*)$.

2. Generalized vectorial quasi-equilibrium problems

Theorem 2.1 *Let $(E, \{C_K\})$ be a W-space, $X \subset E$ be a nonempty weakly W-convex subset, Z be a topological space, $C{:}X \to 2^Z$ be a family of domination structure in Z such that $\mathrm{int}C(x) \neq \emptyset$, $F : X \times X \to 2^Z$, $M : X \to 2^Z$ be two set valued maps, and satisfy the following conditions:*

(i) The map M is nonempty open-valued and W-convex valued, and M is a lower semi-continuous map;

(ii) $\nabla = \{x \in X : M(x) \cap Q(x) \neq \emptyset\}$ is a closed set, where

$$Q(x) = \{y \in X : (x, y) \notin F_x^{-1}(C)\}, \forall x \in X;$$

(iii) For all $x \in X$, $Q(x)$ is a W-convex set;

(iv) $F_x^{-1}(C)$ is a closed set;

(v) $\forall x \in \nabla, (x, x) \in F_x^{-1}(C)$;

(vi) M is transfer closed-valued on $X \backslash \nabla, M \cap Q$ is transfer closed-valued on ∇;

(vii) $\forall y \in X$, for any finite subset $K \subset X$ and any $x \in C_K \backslash K, M(x) \cap Q(x) \neq \emptyset$ and $y \notin M(x) \cap Q(x)$;

(viii) There exists $\hat{x} \in X$, such that if $\hat{x} \in \nabla$, then $\mathrm{Cl}(M(\hat{x}))$ is a compact set, if $\hat{x} \in X \backslash \nabla$, then $\mathrm{Cl}((M(\hat{x}) \cap Q(\hat{x}))$ is a compact set.

Then there exists $x^* \in X$ such that $x^* \in M(x^*)$ and $(x^*, y) \in F_{x^*}^{-1}(C)$ for all $y \in M(x^*)$.

Proof $(X, \{C_{K \cap X} \cap X\})$ is a W-space from X is a weakly W-convex set, define set-valued map $T : X \to 2^X$ as $T(x) = M(x) \cap Q(x)$, $\forall x \in X$, then the Graph of Q is an open set from (iv), so T is lower-semicontinuous from (i) and lemma 4.2 in [12], for all $x \in X$, $M(x)$ and $T(x)$ be open and W-convex sets from above and (i), define set-valued map $S : X \to 2^X$ as: $S(x) = \begin{cases} T(x), & x \in \nabla \\ M(x), & x \in X \backslash \nabla \end{cases}$, then $\forall x \in X$, the set $S(x)$ is a nonempty open and W-convex subset, we can prove that S is a lower-semicontinuous map, in fact, for any open set $V \subset X$, the set

$$\begin{aligned} \{x \in X : S(x) \cap V \neq \emptyset\} &= \{x \in \nabla : T(x) \cap V \neq \emptyset\} \cup \{x \in X \backslash \nabla : M(x) \cap V \neq \emptyset\} \\ &= \{x \in X : T(x) \cap V \neq \emptyset\} \cup ((X \backslash \nabla) \cap \{x \in X : M(x) \cap V \neq \emptyset\}) \end{aligned}$$

is an open set from the lower-semicontinuous of M and T and the condition (ii) and lemma 1.1, hence S is a lower-semicontinuous map, so the condition (i) of lemma 1.2 is satisfied. we can prove for all $y \in X, X \backslash S^{-1}(y)$ is a W-convex set, in fact, if there exists $y \in X$, such that $X \backslash S^{-1}(y)$ isn't a W-convex set, then there exists finite subset $K \subset X \backslash S^{-1}(y)$ and there exists $x \in C_K \backslash K$ such that $x \notin X \backslash S^{-1}(y)$, i.e., $y \in S(x)$, by condition (vii) we have $M(x) \cap Q(x) \neq \emptyset$ and $y \notin M(x) \cap Q(x)$, so $y \notin S(x)$, contradict to $y \in S(x)$, hence $X \backslash S^{-1}(y)$ is W-convex, so the condition (ii) of lemma 1.2 is satisfied. We can prove S is transfer closed valued, in fact, $\forall x \in X$, if $y \notin S(x)$, if $x \in \nabla$ then

$$y \notin S(x) = (M \cap Q)(x),$$

hence there exists $x_0 \in \nabla$ such that

$$y \notin \mathrm{Cl}(M \cap Q)(x_0) = \mathrm{Cl}(S(x_0))$$

from the transfer closedness of $M \cap Q$, for the same reason if $x \in X \backslash \nabla$, there exists $x_0 \in X \backslash \nabla$, such that

$$y \notin \mathrm{Cl}(M(x_0)) = \mathrm{Cl}(S(x_0)),$$

hence $\forall x \in X$, if $y \notin S(x)$ then there exists $x_0 \in X$ such that $y \notin \mathrm{Cl}(S(x_0))$, i.e., S is transfer closed valued. there exists $\hat{x} \in X$ such that $\mathrm{Cl}(S(\hat{x}))$ be a compact set from (viii), so the condition (iii) of lemma 1.2 is satisfied , by lemma 1.2, there exists $x^* \in X$ such that $x^* \in S(x^*), \forall x \in \nabla$, $x \notin Q(x)$ from (v), then $\forall x \in \nabla$, $x \notin T(x)$, hence $x^* \notin \nabla$, by the definition of S we have $x^* \in M(x^*)$ and $M(x^*) \cap Q(x^*) = \emptyset$, i.e., $x^* \in M(x^*)$ and $(x^*, y) \in F_{x^*}^{-1}(C)$, $\forall y \in M(x^*)$. □

Theorem 2.1 generalizes and improves the existence of generalized vector equilibrium problem in [5] from H-space to W-space. If we choose

$$C(x) = Z\backslash - \mathrm{int}P(x) \text{ and } F_x^{-1} = F_x^-,$$

then

$$(x,y) \in F_x^{-1}(C) \Leftrightarrow F(x,y) \not\subset -\mathrm{int}P(x), (x,y) \notin F_x^{-1}(C) \Leftrightarrow F(x,y) \subset -\mathrm{int}P(x).$$

So from theorem 2.1, we have corollary 2.2, which is the existence theorem of generalized set-valued quasi-equilibrium problem, and the generalized set-valued equilibrium problem in [6] is special case of generalized set-valued quasi-equilibrium problem ; if we choose $C(x) = Z\backslash - \mathrm{int}P$ and $F_x^{-1} = F_x^-$, we can easily get the existence theorem of another kind of set-valued quasi-equilibrium problem which is the generalization of set-valued equilibrium problem in [5], from theorem 2.1, we can also get the existence theorems of scalar quasi-equilibrium problem and vector quasi-equilibrium problem on W-space which are the generalization of quasi-equilibrium problem in [1–4], and we omit these results here.

Corollary 2.2 *Let $(E, \{C_K\})$ be a W-space, $X \subset E$ be a nonempty weakly W-convex subset, Z be a topological space, $P : X \to 2^Z$ be a family of domination structure in Z such that $\mathrm{int}P(x) \neq \emptyset$, $F : X \times X \to 2^Z$, $M : X \to 2^Z$ be two set valued maps and satisfy the following conditions:*

(i) M is a nonempty open-valued and W-convex valued and lower semi-continuous map;

(ii) $\nabla = \{x \in X : M(x) \cap Q(x) \neq \emptyset\}$ is a closed set, where

$$Q(x) = \{y \in X : F(x,y) \subset -\mathrm{int}P(x)\}, \forall x \in X;$$

(iii) For all $x \in X, Q(x)$ is a W-convex set;

(iv) $\{(x,y) \in X \times X : F(x,y) \subset -\mathrm{int}P(x)\}$ is a open set;

(v) $\forall x \in \nabla$, $F(x,x) \not\subset -\mathrm{int}P(x)$;

(vi) M is transfer closed-valued on $X\backslash\nabla$; $M \cap Q$ is transfer closed-valued on ∇;

(vii) $\forall y \in X$, for any finite subset $K \subset X$ and any $x \in C_K\backslash K, M(x) \cap Q(x) \neq \emptyset$ and $y \notin M(x) \cap Q(x)$;

(viii) There exists $\hat{x} \in X$, if $\hat{x} \in \nabla$, then $\mathrm{Cl}(M(\hat{x}))$ be a compact set, if $\hat{x} \in X\backslash\nabla$ then $\mathrm{Cl}((M(\hat{x}) \cap Q(\hat{x}))$ be a compact set.

Then there exists $x^* \in X$ such that $x^* \in M(x^*)$ and $F(x^*,y) \not\subset -\mathrm{int}P(x^*)$ for all $y \in M(x^*)$.

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W- 空间上的广义向量拟平衡问题

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摘 要: 引入了广义向量拟平衡问题和广义集值拟平衡问题，得到了W - 空间上广义向量拟平衡问题的存在性定理，作为推论，得到了W - 空间上四类拟平衡问题的存在性定理，这些结果推广和改进了文献 [1–7] 的相应结果.