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Products of Pointwise Pseudo-Quasi-Metrics on Lattices *

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Abstract: In [6,11], A theory of pointwise pseudo-quasi-metrics was based on completely distributive lattices. In this paper a product pointwise p.q. metric function is constructed on the product of countably many molecular lattices by distance functions. Hence it is proved that countable product of pointwise pseudo-quasi-metric molecular lattices is pointwise pseudo-quasi-metrizable.

Key words: cotopology; topological molecular lattice; pseudo-quasi-metric; generalized order homomorphism.

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1. Introduction

In 1985, Wang introduced a theory of pointwise topology on completely distributive lattices in [10]. Subsequently Fan respectively discussed product operations in the category of molecular lattices and in the category of topological molecular lattices in [1,2]. These provide necessary algebraic methods for further study of the theory of topological molecular lattices. In [3], based on an equivalent characterization of Erceg's pseudo-quasi-metric, Peng and Xu proved that the categorical product of countably many pseudo-quasi-metric molecular lattices is pseudo-quasi-metrizable. In 1988, Yang presented a theory of pseudo-quasi-metrics on completely distributive lattices, but his definition was not completely given by a distance function. In 1996, to reflect the characteristics of pointwise topologies on molecular lattices, Shi introduced a new theory of pointwise quasi-uniformities and a new theory of pointwise pseudo-quasi-metrics on completely distributive lattices (see[6]). Many ideal results in general topology were generalized into topological molecular lattices. Meanwhile Shi proved that the pointwise p.q. metric is equivalent to Yang's p.q. metric. In this paper, our aim is to prove that the categorical product of countably many pointwise pseudo-quasi-metric molecular lattices is pointwise pseudo-quasi-metrizable.

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2. Preliminaries

Throughout this paper, L always denotes a completely distributive lattice with the minimal element 0 and the maximal element 1. $M(L)$ denotes the set of all non-zero $\vee$-irreducible elements in L. For $A \in L$, $\beta^*(A)$ denotes greatest molecular minimal family of A. It is easy to verify that $a \in \beta^*(A)$ if and only if $a \ll A$, where $\ll$ is the waybelow relation[4] in L. Moreover it is an evident fact that $a \in M(L)$ if and only if $\{b \in M(L) \mid b \ll a\}$ is a directed set.

Definition 2.1[6] *A pointwise pseudo-quasi-metric (or p.q. metric for short) on L is a map $d : M(L) \times M(L) \to [0, +\infty)$ satisfying the following conditions (M1)–(M3).*

(M1) $\forall a \in M(L), d(a,a) = 0$;

(M2) $\forall a,b,c \in M(L), d(a,c) \le d(a,b) + d(b,c)$;

(M3) $\forall a,b \in M(L), d(a,b) = \bigwedge_{c \ll b} d(a,c)$.

If d is a pointwise p.q. metric on L, then (L,d) is called a pointwise p.q. metric molecular lattice.

From (M1) and (M3) we can see that $a \le b \Rightarrow d(a,b) = 0$.

Definition 2.2[6] *Let d be a pointwise p.q. metric on L. $\forall r \in (0,+\infty)$, define a map $P_r : M(L) \to L$ by*

$$P_r(a) = \bigvee\{b \in M(L) \mid d(a,b) \ge r\}.$$

Then $\{P_r \mid r \in (0,+\infty)\}$ is called the family of R-neighborhood maps (or R-nbd maps for short) of d.

Theorem 2.3[9] *If d is a pointwise p.q. metric on L, then*

(1) $\{P_r(a) \mid a \in M(L), r \in (0,+\infty)\}$ is a closed base for a cotopology on L. This cotopology is denoted by η_d;

(2) $\{P_r(a) \mid r > 0\}$ is a locally R-neighborhood base at a in the cotopology η_d.

Theorem 2.4[9] *Let $(L_1,d_1),(L_2,d_2)$ be two pointwise p.q. metric molecular lattices, $f : L_1 \to L_2$ be a generalized order homomorphism. Then f is continuous if and only if $\forall a \in M(L_1)$, $\forall \varepsilon > 0$, there exists $\delta > 0$ such that $d_2(f(a),f(b)) < \varepsilon$ whenever $d_1(a,b) < \delta$.*

Theorem 2.5[2] *Let $\{(L_i,\eta_i)\}_{i\in I}$ be a family of topological molecular lattices. Then in the category of topological molecular lattices, $(\bigotimes_i L_i, \eta)$ is their categorical product, where η is the coarsest cotopology making each projective map $p_i : \bigotimes_i L_i \to L_i$ continuous. Further $\{p_i^{-1}(Q) \mid Q \in \eta_i, i \in I\}$ is a subbase of η.*

Other concepts not mentioned in this paper are from [1,2,6,10].

3. Countable product of p.q. metric molecular lattices

The following Theorem is very useful. Its proof is obvious.

Theorem 3.1 *Let (L,d) be a pointwise p.q. metric molecular lattice. Define a map $\rho : M(L) \times M(L) \to [0,+\infty)$ such that $\rho(a,b) = \min\{d(a,b),1\}$. Then*

(1) ρ is a pointwise p.q. metric on L.

(2) d and ρ induce same cotopology, i.e., $\eta_\rho = \eta_d$.

Now we discuss countable product of pointwise p.q. metric molecular lattices.

Since a generalized order homomorphism maps a nonzero $\vee$-irreducible element into a nonzero $\vee$-irreducible element, we can define a pointwise p.q. metric on $\bigotimes_i L_i$ as follows.

Theorem 3.2 *Let $\{(L_i, d_i)\}_{n=1}^{\infty}$ be countably many pointwise p.q. metric molecular lattices, where each d_i is bounded by 1, i.e., $d_i(x,y) \le 1, \forall x, y \in M(L_i)$. We define a map $d : M\left(\bigotimes_i L_i\right) \times M\left(\bigotimes_i L_i\right) \to [0, +\infty)$ by*

$$\forall a, b \in M\left(\bigotimes_i L_i\right), d(a,b) = \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\left(p_i(a), p_i(b)\right).$$

Then

(1) For each i, $d_i\left(p_i(a), p_i(b)\right) = \bigwedge_{c \ll b} d_i\left(p_i(a), p_i(c)\right)$.

(2) d is a pointwise p.q. metric on $\bigotimes_i L_i$. In the sequel, d is called the product of $\{d_i\}_{i=1}^{\infty}$.

Proof (1) For each i and each $c \ll b$, since each p_i preserves the waybelow relation $\ll$, we have that

$$d_i\left(p_i(a), p_i(b)\right) \le d_i\left(p_i(a), p_i(c)\right).$$

This implies that

$$d_i\left(p_i(a), p_i(b)\right) \le \bigwedge_{c \ll b} d_i\left(p_i(a), p_i(c)\right).$$

Moreover $\forall \lambda \ll p_i(b)$, by $p_i(b) = \bigvee_{c \ll b} p_i(c)$ we know that there exists $c \ll b$ such that $\lambda \le p_i(c)$. Thus we have that

$$d_i\left(p_i(a), \lambda\right) \ge d_i\left(p_i(a), p_i(c)\right).$$

This shows that

$$d_i\left(p_i(a), p_i(b)\right) = \bigwedge_{\lambda \ll p_i(b)} d_i\left(p_i(a), \lambda\right) \ge \bigwedge_{c \ll b} d_i\left(p_i(a), p_i(c)\right).$$

(1) is proved.

(2) It is not difficult to check that d satisfies (M1) and (M2) in Definition 2.1. It remains to verify (M3) in Definition 2.1. Take $a, b \in M\left(\bigotimes_i L_i\right)$. By (1) we can obtain that

$$\begin{aligned} d(a,b) &= \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\left(p_i(a), p_i(b)\right) = \sum_{i=1}^{\infty} \frac{1}{2^i} \bigwedge_{c \ll b} d_i\left(p_i(a), p_i(c)\right) \\ &\le \bigwedge_{c \ll b} \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\left(p_i(a), p_i(c)\right) = \bigwedge_{c \ll b} d(a,c). \end{aligned}$$

Now we prove that $d(a,b) \geq \bigwedge_{c \ll b} d(a,c)$.

Suppose that $d(a,b) = r$. Then $\forall \varepsilon > 0$,

$$d(a,b) = \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(b)) = \sum_{i=1}^{\infty} \frac{1}{2^i} \bigwedge_{c \ll b} d_i\,(p_i(a), p_i(c)) < r + \varepsilon.$$

Let $r_i = \frac{1}{2^i} \bigwedge_{c \ll b} d_i\,(p_i(a), p_i(c))$. Then

$$\bigwedge_{c \ll b} d_i\,(p_i(a), p_i(c)) < 2^i r_i + \varepsilon.$$

This implies that there exists $c_i \ll b$ such that

$$d_i\,(p_i(a), p_i(c_i)) < 2^i r_i + \varepsilon.$$

Hence we have that

$$\sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(c_i)) < \sum_{i=1}^{\infty} r_i + \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = r + \varepsilon.$$

Since $\{c \mid c \ll b\}$ is a directed set, we can take $e_i \ll b$ such that $c_1 \leq e_i, c_2 \leq e_i, \cdots, c_i \leq e_i$ for each i. Thus for any natural number n, we obtain that

$$\sum_{i=1}^{n} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(e_n)) + \sum_{i=n+1}^{\infty} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(c_i)) \leq \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(c_i)) < r + \varepsilon.$$

This shows that

$$\sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(e_n)) + \sum_{i=n+1}^{\infty} \frac{1}{2^i}\,[d_i\,(p_i(a), p_i(c_i)) - d_i\,(p_i(a), p_i(e_n))] < r + \varepsilon.$$

Therefore for any natural number n, it follows that

$$\sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(e_n)) < r + \varepsilon - \sum_{i=n+1}^{\infty} \frac{1}{2^i}\,[d_i\,(p_i(a), p_i(c_i)) - d_i\,(p_i(a), p_i(e_n))] \leq r + \varepsilon + \sum_{i=n+1}^{\infty} \frac{1}{2^i}.$$

This implies that

$$\bigwedge_{c \ll b} d(a,c) \leq \bigwedge_{e_n \ll b} d_i\,(a, e_n) \leq \bigwedge_{e_n \ll b} \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\,(p_i(a), p_i(e_n)) \leq r + \varepsilon.$$

Further we have that $\bigwedge_{c \ll b} d(a,c) \leq r = d(a,b)$. So d satisfies (M3). We complete the proof.

Theorem 3.3 *Let $\{(L_i, d_i)\}_{i=1}^{\infty}$ be countably many pointwise p.q. metric molecular lattices, d be the product of $\{d_i\}_{i=1}^{\infty}$. Then $\eta_d = \bigotimes_i \eta_{d_i}$.*

Proof For any $a \in M\left(\bigotimes_i L_i\right)$ and $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2^i}$. Then $d_i(p_i(a), p_i(b)) < \varepsilon$ whenever $d(a,b) = \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\left(p_i(a), p_i(b)\right) < \delta(i = 1, 2, \cdots)$. By Theorem 2.4 we know that each projective map $p_i : \left(\bigotimes_i L_i, \eta_d\right) \to (L_i, \eta_{d_i})$ is continuous. Hence we have that $\bigotimes_i \eta_{d_i} \subseteq \eta_d$.

To prove the inverse inclusion, let $A \in \eta_d$, $a \in M\left(\bigotimes_i L_i\right)$ and $a \not\leq A$. Then there exists $r > 0$ such that $A \leq P_r(a)$. Hence $\forall b \in \beta^*(A)$, we have that

$$d(a,b) = \sum_{i=1}^{\infty} \frac{1}{2^i}\, d_i\left(p_i(a), p_i(b)\right) \geq r.$$

Choose n such that

$$\sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} < \frac{r}{2}\,.$$

Then $\forall b \in \beta^*(A)$ it follows that

$$\sum_{i=1}^{n} \frac{1}{2^i}\, d_i\left(p_i(a), p_i(b)\right) \geq \frac{r}{2}.$$

Suppose that $d_k\left(p_k(a), p_k(b)\right) = \max\{d_i\left(p_i(a), p_i(b)\right)\}$. Then we have that

$$d_k\left(p_k(a), p_k(b)\right) \geq \left(1 - \frac{1}{2^n}\right) d_k\left(p_k(a), p_k(b)\right) \geq \sum_{i=1}^{n} \frac{1}{2^i}\, d_i\left(p_i(a), p_i(b)\right) \geq \frac{r}{2}.$$

This implies that $p_k(b) \leq P_{r/2}\left(p_k(a)\right)$, i.e., $b \leq p_k^{-1}\left(P_{r/2}\left(p_k(a)\right)\right)$. Hence

$$A \leq \bigvee_{i=1}^{n} p_i^{-1}\left(P_{r/2}\left(p_i(a)\right)\right).$$

By Theorem 2.5 we know that a is not an adherence point of A in $\bigotimes_i \eta_{d_i}$. This shows that $A \in \bigotimes_i \eta_{d_i}$. The inverse inclusion is proved.

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格上点式 p.q. 度量的乘性

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摘 要: 本文在可数多个分子格的乘积上构造了一个乘积点式 p.q. 度量函数，从而证明了点式 p.q. 度量分子格的可数积是可点式 p.q. 度量化的.