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Compact Composition Operators Mapping into the $E_0(p,q)$ Spaces *

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Abstract: Composition operators are used to study the $E_0(p,q)$ spaces, which coincide with the space $Q_{q,0}$ for $p=2$ and the little Bloch space $\mathcal{B}_0$ for $p>0$ and $q>1$. The compactness of these operators is also considered. The criteria for these operators to be compact are given in terms of the Carleson measure.

Key words: Compact operator; function space; composition operator; Bloch space; Carleson measure.

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1. Introduction

Let $D=\{z\in D:|z|<1\}$ be the unit disk of complex plane, $H(D)$ be the space of all analytic functions on D. Denote Lebesgue measure on D by dm, normalized so that $m(D)=1$. For $a\in D,\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$ is the Mobius transformation of D to itself and $g(z,a)=\log|\frac{1-\bar{a}z}{a-z}|$ is the Green function of D with singularity at a. Every analytic self-map $\varphi:D\to D$ of the unit disk induces through composition a linear composition operator C_φ from $H(D)$ to itself. It is a well-known consequence of Littlewood's subordination principle that φ induces through composition a bounded linear operator on the classical Hardy and Bergman spaces ([1],[2]). That is, if we define C_φ by $C_\varphi(f)=f\circ\varphi$ for $f\in H(D)$, then $C_\varphi:H^p\to H^p$ and $C_\varphi:A^p\to A^p$ are bounded operators. A problem that has received much attention recently is to relate function theoretic properties of φ to operator theoretic properties of the restriction of C_φ to various Banach spaces of analytic function. In this paper we study this problem in the context of what are known as the $E_0(p,q)$ spaces, which were recently studied by Tan Haiou in [3].

We have $E_0(2,q)=Q_{q,0}$, and specially $E_0(2,1)=\text{VMOA}$ and for $p>0$ and $q>1$, $E_0(p,q)=\mathcal{B}_0$, due to [4]. For $0<q<\infty$, we say that a positive measure μ defined on D

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is a compact q-Carleson measure provided $\mu(S(I)) = o(|I|^q)$ for all subarcs I of ∂D, where $|I|$ denotes the arc length of I and $S(I)$ denotes the usual Carleson box based on I. We denote the set of meromorphic functions on D by M. Aulaskari R. and Zhao R. proved in [5], for $p > 2, q > 1$ and $f \in M$, then $f \in B_0^{\#}(p,q)$ if and only if $d\mu_{f,p,q}$ is a compact q-Carleson measure. Tan H. and Xiao J. proved in [6], for $p > 2, 0 < q \le 1$ and $f \in M$, then $f \in B_0^{\#}(p,q)$ if and only if $d\mu_{f,p,q}$ is a compact q-Carleson measure. If $p = 2, q \ge 1$ and $f \in M$, then $f \in B_0^{\#}(p,q)$ if and only if there is a $\delta \in (0,1)$ such that

$$\lim_{|a|\to 1} \int_{D(a,\delta)} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^q \mathrm{d}\mu_{f,p,q}(z) = 0.$$

where $D(a,\delta) = \{z \in D : |\sigma_a(z)| < \delta\}$. Our work involves hyperbolic version of the $E_0(p,q)$ spaces. These spaces are defined by using the hyperbolic derivative $\frac{|f'(z)|}{1-|f(z)|^2}$ in place of the spherical derivative in the definition of $B_0^{\#}(p,q)$. Defined the hyperbolic $E_0^h(p,q)$ as follows

$$E_0^h(p,q) = \{f : f \in H(D) \ \text{ and } \ \lim_{|a|\to 1} \int_D \frac{|f'(z)|^p}{(1-|f(z)|^2)^p}(1-|z|^2)^{p-2} g^q(z,a)\mathrm{d}m(z) = 0\}.$$

For φ an analytic self-map of D, we define

$$\mathrm{d}\mu^h_{\varphi,p,q}(z) = \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p+q-2}\mathrm{d}m(z),$$

$$\mathrm{d}\lambda^h_{\varphi,p,q}(z) = \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2} g^q(z,0)\mathrm{d}m(z).$$

As usually, the letter $\mathcal{B}$ denotes Bloch space consisting of those functions $f \in H(D)$ such that $\|f\|_{\mathcal{B}} = \sup_{z\in D}(1-|z|^2)|f'(z)| < \infty$.

Our main result is the following :

Theorem *Suppose $p > 0, q > 0$, $p + q > 1$, and φ is an analytic self-map of D. Then the following statements are equivalent :*

(1) $C_\varphi : \mathcal{B} \to E_0(p,q)$ is bounded.
(2) $C_\varphi : \mathcal{B} \to E_0(p,q)$ is compact.
(3) $\varphi \in E_0^h(p,q)$.
(4) $\mathrm{d}\mu^h_{\varphi,p,q}(z)$ is a compact q-Carleson measure.
(5) $\mathrm{d}\lambda^h_{\varphi,p,q}(z)$ is a compact q-Carleson measure.
(6) $\lim_{|a|\to 1} \int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p+q-2}|\sigma_a'(z)|^q dm(z) = 0$.

Using our result, we can get the following corollary, which is Theorem 1.6 in [7].

Corollary *Let $0 < q < \infty$ and suppose φ is an analytic self-map of D. Then the following statements are equivalent :*

(1) $\varphi \in Q^h_{q,0}$;
(2) $C_\varphi : \mathcal{B} \to Q_{q,0}$ is bounded;

(3) $C_\varphi : \mathcal{B} \to Q_{q,0}$ *is compact;*

(4) $\mathrm{d}\mu_\varphi^h$ *is a compact q-Carleson measure.*

The approach to the problem considered in this paper comes from Smith's work ([7]) on the compactness problem for $C_\varphi : \mathcal{B} \to Q_{q,0}$. Throughout this paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables.

2. Some Lemmas

Here we collect some lemmas which will be used in the main results.

Lemma 1 *If $f \in \mathcal{B}$, then*

$$|f(z)| \le |f(0)| + \frac{\|f\|_{\mathcal{B}}}{2} \log \frac{1+|z|}{1-|z|}, \quad \forall z \in D.$$

Lemma 2[8] For $0 < q < \infty$, *a positive measure μ on D is a compact q-Carleson measure if and only if*

$$\lim_{|a|\to 1} \int_D \left(\frac{1-|a|^2}{|1-\bar{a}z|^2} \right)^q \mathrm{d}\mu(z) = 0.$$

Lemma 3[9] *There exist $f_1, f_2 \in \mathcal{B}$ such that*

$$|f_1'(z)| + |f_2'(z)| \ge \frac{1}{1-|z|^2}, \quad \forall z \in D.$$

Lemma 4[10] *For $0 < r < 1$ and $0 < k < \infty$, then*

$$I(r,k) = 2\pi \int_0^r t(1-t^2)^{-2} (\frac{1}{t})^k \mathrm{d}t < \infty.$$

Lemma 5[3] *If $\varphi \in H(D)$, $p > 0, q > 0$ and $p+q > 1$, then $\varphi \in E_0(p,q)$ if and only if*

$$\lim_{|a|\to 1} \int_D |\varphi'(z)|^p (1-|z|^2)^{p-2} (g(z,a))^q \mathrm{d}m(z) = 0.$$

3. The proofs of main result

In this section we prove our main theorem, the proofs of the theorem are based on above several lemmas.

Proof (3) $\to$ (4). If $\varphi \in E_0^h(p,q)$, then applying the equality $1-|\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}$, we have

$$\begin{aligned} &\lim_{|a|\to 1} \int_D \left(\frac{1-|a|^2}{|1-\bar{a}z|^2} \right)^q \mathrm{d}\mu_{\varphi,p,q}^h(z) \\ &= \lim_{|a|\to 1} \int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p} (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^q \mathrm{d}m(z) \\ &\le C \lim_{|a|\to 1} \int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p} (1-|z|^2)^{p-2} g^q(z,a) \mathrm{d}m(z) = 0. \end{aligned}$$

By Lemma 2, we show that $d\mu^h_{\varphi,p,q}(z)$ is a compact q-Carleson measure.

(4) $\to$ (2). If $d\mu^h_{\varphi,p,q}(z)$ is a compact q-Carleson measure, then $d\mu^h_{\varphi,p,q}(z)$ is a bounded q-Carleson measure. It is easy to show that $C_\varphi : \mathcal{B} \to E(p,q)$ is bounded. For $f \in \mathcal{B}$, we have $C_\varphi f \in E(p,q)$. Since $|(f\circ\varphi)'(z)| \le \|f\|_{\mathcal{B}} \frac{|\varphi'(z)|}{1-|\varphi(z)|^2}$, Lemma 2 implies that

$$\begin{aligned}&\lim_{|a|\to 1}\int_D |(f\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q dm(z)\\ &\le \|f\|^p_{\mathcal{B}} \lim_{|a|\to 1}\int_D \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^q d\mu^h_{\varphi,p,q}(z) = 0.\end{aligned}$$

Thus $C_\varphi f \in E_0(p,q)$, so that $C_\varphi : \mathcal{B} \to E_0(p,q)$ is bounded. To show this operator is compact, we let $\{f_n\} \subset \mathcal{B}$ be such that $\|f_n\|_{\mathcal{B}} \le 1$ for $n \ge 1$, we must show that $\{C_\varphi f_n\}$ has a subsequence that converges in $E_0(p,q)$. By Lemma 1 there is a subsequence of $\{f_n\}$ that converges uniformly on compact subsets of D to an analytic function f. By passing to a subsequence, we may assume that the sequence $\{f_n\}$ itself converges uniformly on compact subset of D to $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} \le 1$. Thus $C_\varphi f \in E_0(p,q)$. Set

$$\|f\|^p_{p,q} = \sup_{a\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q dm(z).$$

To complete the proof, it suffices to show that $\lim_{n\to\infty}\|C_\varphi f_n - C_\varphi f\|_{p,q} = 0$. $\forall \varepsilon > 0$, by $(*)$ and Lemma 2, we can choose $r \in (0,1)$ such that when $r < |a| < 1$

$$\begin{aligned}&\int_D |(f_n\circ\varphi)'(z) - (f\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}\left(1-|\sigma_a(z)|^2\right)^q dm(z)\\ &\le C\int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}\left(1-|\sigma_a(z)|^2\right)^q dm(z) < C\varepsilon,\end{aligned}$$

thus

$$\sup_{r<|a|<1}\int_D |(f_n\circ\varphi)'(z) - (f\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}\left(1-|\sigma_a(z)|^2\right)^q dm(z) < C\varepsilon.$$

Since $\{a : |a| \le r\}$ is a compact set, there is a $t_0 \in (0,1)$ such that, uniformly in n

$$\begin{aligned}&\sup_{|a|\le r}\int_{D-t_0D} |(f_n\circ\varphi)'(z) - (f\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q dm(z)\\ &\le C\sup_{|a|\le r}\int_{D-t_0D} \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q dm(z) < C\varepsilon.\end{aligned}$$

Also, by the uniformly converges of $\{(f_n'\circ\varphi - f'\circ\varphi)(1-|\varphi|^2)\}$ to 0 on compact subsets of D , there exists N_0 such that

$$\begin{aligned}&\int_{t_0D} |(f_n\circ\varphi)'(z) - (f\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q dm(z)\\ &\le C\varepsilon\int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q dm(z) < C\varepsilon,\end{aligned}$$

provided $n > N_0$. Thus for any such n, we have

$$\sup_{|a|\leq r}\int_{t_0 D}|(f_n\circ\varphi)'(z)-(f\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^q\mathrm{d}m(z) < C\varepsilon.$$

Hence, for any $\varepsilon > 0$ there is a N_0 such that $\|C_\varphi f_n - C_\varphi f\|_{p,q} < C\varepsilon, \forall n > N_0$, as required, and the proof is complete.

(2) $\to$ (1). It is easy.

(1) $\to$ (3). Let $f_1, f_2 \in \mathcal{B}$, be the two functions from Lemma 3, then

$$\frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2))^p} \leq C(|(f_1\circ\varphi)'(z)|^p + |(f_2\circ\varphi)'(z)|^p), \forall z\in D,$$

by Lemma 5, we have

$$\begin{aligned}&\lim_{|a|\to 1}\int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}g^q(z,a)\mathrm{d}m(z)\\ &\leq C\lim_{|a|\to 1}\int_D |(f_1\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}g^q(z,a)\mathrm{d}m(z)+\\ &\quad C\lim_{|a|\to 1}\int_D |(f_2\circ\varphi)'(z)|^p(1-|z|^2)^{p-2}g^q(z,a)\mathrm{d}m(z) = 0.\end{aligned}$$

It is easy to see that if $C_\varphi : \mathcal{B}\to E_0(p,q)$ is bounded, then $\varphi\in E_0^h(p,q)$.

(5) $\to$ (4). Since $1-|z|^2 \leq 2\log\frac{1}{|z|}$, we have

$$(1-|z|^2)^q \leq 2^q(\log\frac{1}{|z|})^q = 2^q g^q(z,0),$$

hence

$$\begin{aligned}&\lim_{|a|\to 1}\int_D\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^q\frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p+q-2}\mathrm{d}m(z)\\ &\leq C\lim_{|a|\to 1}\int_D\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^q\frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}g^q(z,0)\mathrm{d}m(z).\end{aligned}$$

(4) $\to$ (5). For $a\in D$, we write

$$\int_D\frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}g^q(z,0)\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^q\mathrm{d}m(z) = I_1(a)+I_2(a).$$

Because $\frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(\frac{1-|a|^2}{|1-\overline{a}z|^2})^q$ is subharmonic in z, using Lemma 4, we have

$$\begin{aligned}
I_1(a) &= \int_{|z|\le 1/4} \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}(\log\frac{1}{|z|})^q \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}m(z) \\
&\le \sup_{|z|\le 1/4}\left\{\frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}\left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q\right\}\int_{|z|\le 1/4}(\log\frac{1}{|z|})^q(1-|z|^2)^{p-2}\mathrm{d}m(z) \\
&\le C\sup_{|z|\le 1/4}\int_{|u-z|<1/4}\frac{|\varphi'(u)|^p}{(1-|\varphi(u)|^2)^p}(1-|u|^2)^{p+q-2}\left(\frac{1-|a|^2}{|1-\overline{a}u|^2}\right)^q \mathrm{d}m(u) \\
&\le C\int_D \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}\mu^h_{\varphi,p,q}(z).
\end{aligned}$$

Since $|z|\in(1/4,1)$, $\log\frac{1}{|z|}\le 8(1-|z|^2)$, therefore

$$\begin{aligned}
I_2(a) &= \int_{\frac14<|z|<1} \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}(\log\frac{1}{|z|})^q \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}m(z) \\
&\le C\int_{\frac14<|z|<1} \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p+q-2}\left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}m(z) \\
&\le C\int_D \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}\mu^h_{\varphi,p,q}(z),
\end{aligned}$$

and so

$$\int_D \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}\lambda^h_{\varphi,p,q}(z)\le C\int_D \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^q \mathrm{d}\mu^h_{\varphi,p,q}(z). \tag{1}$$

The implication (4) $\to$ (5) follows .

(3) $\leftrightarrow$ (6). Since $|\sigma_a'(z)|(1-|z|^2)=1-|\sigma_a(z)|^2\le 2\log\frac{1}{|\sigma_a(z)|}=2g(z,a)$, (3) $\to$ (6) is true. Conversely, set $a=0$ in (1), we have

$$\int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p-2}g(z,0)^q\mathrm{d}m(z)\le C\int_D \frac{|\varphi'(z)|^p}{(1-|\varphi(z)|^2)^p}(1-|z|^2)^{p+q-2}\mathrm{d}m(z),$$

substitute $\varphi(z)$ by $\varphi\circ\sigma_a(z)$ we get

$$\begin{aligned}
&\int_D \frac{|\varphi'(\sigma_a(z))|^p}{(1-|\varphi(\sigma_a(z))|^2)^p}|\sigma_a'(z)|^p(1-|z|^2)^{p-2}g(z,0)^q\mathrm{d}m(z) \\
&\le C\int_D \frac{|\varphi'(\sigma_a(z))|^p}{(1-|\varphi(\sigma_a(z))|^2)^p}|\sigma_a'(z)|^p(1-|z|^2)^{p+q-2}\mathrm{d}m(z),
\end{aligned}$$

let $z=\sigma_a(u)$, we obtain

$$\begin{aligned}
&\int_D \frac{|\varphi'(u)|^p}{(1-|\varphi(u)|^2)^p}(1-|u|^2)^{p-2}g(u,a)^q\mathrm{d}m(u) \\
&\le C\int_D \frac{|\varphi'(u)|^p}{(1-|\varphi(u)|^2)^p}(1-|u|^2)^{p+q-2}|\sigma_a'(u)|^q\mathrm{d}m(u),
\end{aligned}$$

we see that (6) → (3) is also true.

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映入 $E_0(p,q)$ 的紧复合算子

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摘 要: 应用复合算子研究 $E_0(p,q)$ 空间，当 $p=2$ 时，它就是 $Q_{q,0}$，当 $p>0$ 且 $q>1$ 时，它就是小 Bloch 空间 $\mathcal{B}_0$. 讨论了复合算子的紧性并利用 Carleson 测度给出复合算子是紧的判别准则.