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Blow-up Estimates for a Non-Newtonian Filtration Equation *

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Abstract: In this paper, blow-up estimates for a class of quasilinear reaction-diffusion equations(non-Newtonian filtration equations) in term of the nonexistence result for quasilinear ordinary differential equations are established to extends the result for semi-linear reaction-diffusion equations(Newtonian filtration equations).

Key words: blow up estimate; non-existence; quasilinear reaction-diffusion equation.

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The purpose of this paper is to derive a bound for the rate of blow-up of solutions to the Non-Newtonian filtration equation

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + u^m - u^{p-1}, \tag{1}$$

or

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + u^m, \tag*{(1)$'$}$$

where $u \geq 0, m > p-1, p \geq 2$. The blow up rate estimates of positive radial solutions were established by Weissler in [1] for the (1) or (1)' with $p=2$. In this paper, we get the same result for the (1) or (1)' with $p \geq 2$. Then, it extents and complement the result in [1].

This problem appears in the study of non-Newtonian fluids([2,3]) and nonlinear filtration theory ([4]). In the non-Newtonian fluids theory, the quantity p is a characteristic of the madium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

Let $B(\rho)$ denote the open ball in $\mathbf{R}^N$ of radius ρ, center at 0. Also, for $T > 0$, let $\Gamma = \Gamma(\rho, T) = B(\rho) \times (0,T) \subset \mathbf{R}^{N+1}$. A typical point in Γ is denoted by (x,t), with $x \in B(\rho)$ and $t \in (0,T)$.

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Theorem 1 *Suppose for $\rho > 0$ and $T > 0$ the function $u : \Gamma(\rho, T) \to \mathbf{R}$ satisfies:*

(a) $u \in C^1(\Gamma)$ and u has continuous second order x-derivatives throughout Γ;

(b) $u \geq 0$ and $u_t \geq 0$ in Γ;

(c) for each $t \in (0, T), u(\cdot, t)$ is radially symmetric and nonincreasing as a function $r = |x|$;

(d) for each $t \in (0, T), u_t(\cdot, t)$ achieves its maximum at $x = 0$;

(e) u satisfies (1) or (1)' throughout Γ;

(f) $u(0, t) \to \infty$ as $t \to T$.

Assume that $N = 2, p \geq 2, m \geq 0$ or $N > p, p \geq 2, p - 1 < m < \frac{N(p-1)+p}{N-p}$. Then there exists a constant $C > 0$ such that

$$u(x, t) \leq C(T - t)^{-1/(m-1)} \tag{2}$$

for all $(x, t) \in \Gamma$.

To prove the main Theorem 1, we give the following lemma

Lemma 1 *Let $m > p - 1, p \geq 2$ and $N > 2$, and suppose $N/p < (m+1)/(m-p+1)$(that is $m < \frac{N(p-1)+p}{N-p}$), then there does not exist a positive C^1 function $v(r) : [0, \infty) \to \mathbf{R}$ with $v'(0) = 0$ and*

$$(|v'|^{p-2}v')' + \frac{N-1}{r}|v'|^{p-2}v' + v^m(r) = 0, \quad r > 0. \tag{3}$$

Proof Suppose there exists such a function v, then

$$(r^{N-1}\Phi_p(v'))' + r^{N-1}v^m(r) = 0,$$

and

$$r^{N-1}\Phi_p(v')(r) = -\int_0^r s^{N-1}v^m(s)\mathrm{d}s. \tag{4}$$

where $\Phi_p(v) = |v|^{p-2}v$. We first dispense with the case $N \leq p$. Using (4), we see that if $r \geq 1$, then

$$v'(r) \leq -C^{1/(p-1)}r^{(1-N)/(p-1)}$$

for some $C > 0$. Integrating, we get

$$v(r) \leq v(1) + C^{1/(p-1)}(p-1)/(N-p)(r^{(p-N)/(p-1)} - 1).$$

and so $v(r) \to -\infty$ as $r \to \infty$. This contradicts $v(r) > 0$ and proves the lemma 1 for $N \leq p$.

Now suppose $N > p$. Formula (4) implies that $v(r)$ is decresing and therefore that

$$-r^{N-1}\Phi_p(v') = \int_0^r s^{N-1}v^m(s)\mathrm{d}s \geq r^N v^m(r)/N,$$

or

$$v'(r) \leq -(1/N)^{1/(p-1)}r^{1/(p-1)}v^{m/(p-1)}(r).$$

This inequality is easily integrated to give

$$v^{(m-p+1)/(p-1)} \leq p/(m-p+1)N^{1/(p-1)}r^{-p/(p-1)}.$$

In particular,

$$\lim_{r\to+\infty}\sup r^{p/(m-p+1)}v(r) < +\infty. \tag{5}$$

At this point we use the hypothesis that $N/p < (m+1)/(m-p+1)$. This, along with (5) implies that

$$\int_0^{+\infty} r^{N-1}v^{m+1}(r)\mathrm{d}r < +\infty. \tag{6}$$

We multiply (3) by $r^{N-1}v(r)$ and use the identity

$$(r^{N-1}\Phi_p(v')v)' = (N-1)r^{N-2}\Phi_p(v')v + r^{N-1}(\Phi_p(v'))'v + r^{N-1}|v'|^p.$$

This gives

$$(r^{N-1}\Phi_p(v')v)' - r^{N-1}|v'|^p + r^{N-1}v^{m+1} = 0.$$

Integrating from 0 to r we get

$$-r^{N-1}\Phi_p(v')v(r) + \int_0^r s^{N-1}|v'(s)|^p\mathrm{d}s = \int_0^r s^{N-1}v^{m+1}(s)\mathrm{d}s. \tag{7}$$

Since $v(r) > 0$ and $v'(r) < 0$, formulas (6) and (7) imply

$$\int_0^{+\infty} s^{N-1}|v'|^p\mathrm{d}s \le \int_0^{+\infty} s^{N-1}v^{m+1}(s)\mathrm{d}s < +\infty. \tag{8}$$

We multiply (3) by $r^N v'(r)$ and use the identities

$$(r^N|v'|^p)' = Nr^{N-1}|v'|^p + pr^N|v'|^{p-1}v'',$$
$$(r^Nv^{m+p-1})' = Nr^{N-1}v^{m+p-1} + (m+p-1)r^Nv^{m+p-2}v'.$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^N|v'|^p/p + \frac{r^Nv^{m+p-1}}{m+p-1}) = \frac{N}{(m+p-1)}r^{N-1}v^{m+p-1} +$$
$$\frac{N}{p}r^{N-1}|v'|^p + r^Nv^{m+p-2}v' + 1/(p-1)(-(N-1)r^{N-1}|v'|^p - r^Nv^mv').$$

Integrating from 0 to x we get

$$\frac{r^N|v'|^p}{p} + \frac{r^Nv^{m+p-1}(r)}{m+p-1} = \frac{N}{(m+p-1)}\int_0^r s^{N-1}v^{m+p-1}\mathrm{d}s +$$
$$(\frac{N}{p} - \frac{N-1}{p-1})\int_0^s s^{N-1}|v'|^p\mathrm{d}s + \int_0^r s^Nv^{m+p-2}v'\mathrm{d}s - 1/(p-1)\int_0^r s^Nv^mv'\mathrm{d}s,$$

then

$$\frac{r^N|v'|^p}{p} + \frac{r^N}{(p-1)(m+1)}v^{m+1}(r)$$
$$= (\frac{N}{p} - \frac{N-1}{p-1})\int_0^r s^{N-1}|v'|^p\mathrm{d}s + \frac{N}{(p-1)(m+1)}\int_0^r s^{N-1}v^{m+1}(s)\mathrm{d}s. \tag{9}$$

Let $h(r) = r^N|v'|^p/p + \frac{r^N}{(p-1)(m+1)}v^{m+1}(r)$, by (8) and (9) we see $\lim_{x\to\infty} h(x) = l$ exists. Furthermore, again by virtue of (8), we have that $\int_0^\infty t^{-1}h(t)ds < +\infty$; and so $l = 0$. Thus letting $r \to +\infty$ in (9) yields

$$\frac{N}{(p-1)(m+1)}\int_0^{+\infty} s^{N-1}v^{m+1}(s)\mathrm{d}s = (\frac{N-1}{p-1} - \frac{N}{p})\int_0^{+\infty} s^{N-1}|v'|^p\mathrm{d}s.$$

Finally, (6) and (8) together imply

$$N/p \geq (m+1)/(m-p+1).$$

This contradicts the hypothesis that $N/p < (m+1)/(m-p+1)$, and thereby proves the Lemma 1.

The proof of Theorem 1 is based upon modification of methods of Weissler [1] used to prove Theorem.

Proof of the Theorem 1 We consider equation (1)(Eq.(1)' being similar). For $0 < t < T$, let $\alpha(t) = u(0,t)^{(m-p+1)/p}$; then $\alpha(t) \to \infty$ as $t \to T$. For $t \in (0,T)$ and $y \in B(\rho\alpha(t))$, let

$$v(y,t) = \frac{u(y/\alpha(t),t)}{u(0,t)}.$$

Since $0 \leq u(x,t) \leq u(0,t)$, it follows that

$$0 \leq v(y,t) \leq 1. \tag{10}$$

Furthermore, a routine calculation shows that

$$\mathrm{div}(|\nabla v|^{p-2}\nabla v) + v^m(y,t) = \frac{[u_t(y/\alpha(t),t) + u^{p-1}(y/\alpha(t),t)]}{u^m(0,t)}.$$

Hypotheses (b) and (d) therefore imply that

$$0 \leq \mathrm{div}(|\nabla v|^{p-2}\nabla v) + v^m(y,t) \leq (u_t(0,t) + u^{p-1}(0,t))/u^m(0,t). \tag{11}$$

Since $u(\cdot,t)$ is radially symmetric, the same is true for $v(\cdot,t)$; and thus we may set

$$v(y,t) = w(r,t),$$

where $|y| = r$ and $0 \leq r < \rho\alpha(t)$. Note that for each $t \in (0,T), w(\cdot,t)$ is a C^1 function on $[0,\rho\alpha(t)]$ with $w(0,t) = 1$ and $w_r(0,t) = 0$. Rewriting (10) and (11) in terms of w, we get

$$0 \leq w(r,t) \leq 1, \tag{12}$$

$$0 \leq (\Phi_p(w_r))_r + (N-1)/r\Phi_p(w_r) + w^m \leq \frac{u_t(0,t) + u^{p-1}(0,t)}{u^m(0,t)}, \tag{13}$$

where w_r the derivative of w with repect r. Furthermore, $w_r \leq 0$ by hypothesis (c), and so (13) implies

$$(\Phi_p(w_r))_r w_r + (N-1)/r|w_r|^p + w^m w_r \leq 0,$$

which in turn says that

$$\frac{\partial}{\partial r}((p-1)/p|w_r|^p + w^{m+1}/(m+1)) \leq -(N-1)/r|w_r|^p \leq 0.$$

Integrating this last inequality from 0 to r shows that

$$(p-1)/p|w_r|^p + w^{m+1}/(m+1) \leq 1/(m+1),$$

and thus

$$|w_r(r,t)| \leq (\frac{p}{(p-1)(m+1)})^{1/p}. \tag{14}$$

We now claim that

$$\liminf_{t \to T} \frac{u_t(0,t)}{u^m(0,t)} > 0. \tag{15}$$

We proceed by contradiction as in [1]. Suppose t_n is a sequence in $(0,T)$ with $t_n \to T$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{u_t(0,t_n)}{u^m(0,t_n)} = 0. \tag{16}$$

By using the Ascoli-Alzela theorem, we know that there is a subsequence, which is still called t_n, and a function $\overline{w} \in C([0,\infty))$ such that $w(\cdot,t_n) \to \overline{w}$ uniformly on compact subsets of $[0,\infty)$. In particular, because of the properties of each $w(\cdot,t_n)$, we know that $\overline{w} \geq 0, \overline{w}(0) = 1$, and $\overline{w}$ is nonincreasing on $[0,\infty)$. Moreover, formula (14) implies that each $w(\cdot,t_n)$ is Lipschitz with a Lipschitz constant of $(\frac{p}{(p-1)(m+1)})^{1/p}$. The same is therefore true of $\overline{w}$, and so $\overline{w}$ is absolutely continuous on $[0,\infty)$. Next we consider $w(\cdot,t_n)$ and $\overline{w}$ as distributions on $(0,\infty)$. (Let $w(r,t_n) = 0$ for $r \geq \rho\alpha(t_n)$.) Clearly, $w(\cdot,t_n) \to \overline{w}$ in the sence of distributions; and hence

$$w_r(\cdot,t_n) \to \overline{w}_r, \quad (\Phi_p(w_r))_r(\cdot,t_n) \to (\Phi_p(\overline{w}_r))_r,$$

in the sence of distributions. Thus, formulas (13) and (16) imply that

$$(\Phi_p(\overline{w}_r))_r + (N-1)/r\Phi_p(\overline{w}_r) + \overline{w}^m = 0, \tag{17}$$

as distributions on $(0,\infty)$. This can be rewritten as

$$(r^{N-1}\Phi_p(\overline{w}_r))_r + r^{N-1}\overline{w}^m = 0. \tag{18}$$

Since $\overline{w}$ is absolutely continuous, it follows immediately from (17) that $\overline{w}$ is C^1 on $(0,\infty)$. In particular, since $\overline{w} \geq 0$, the local existence and uniqueness of C^1 solutions of (18) on $(0,\infty)$ guarantee that $\overline{w} > 0$ on $(0,\infty)$.

If $N = 2, p > 2$, we proceed as follows. From Eq (18), we infer that $r\Phi_p(\overline{w}_r)$ are decreasing, and that there exist $M < 0$ and $r_0 > 0$ such that

$$r\Phi_p(\overline{w}_r) < M \quad \text{for } r \in (r_0, +\infty).$$

The last inequality implies that

$$\overline{w}(s) > \overline{w}(s) - \overline{w}(t) > (-M)^{1/(p-1)} \int_s^t r^{-1/(p-1)} \mathrm{d}r$$
$$= (-M)^{1/(p-1)} (t^{(p-2)/(p-1)} - s^{(p-2)/(p-1)}) \tag{19}$$

for $r_0 \leq s \leq t$. Letting $t \to +\infty$ in (19), we obtain a contradiction.

If $N = 2, p = 2$, a similar argument implies that

$$\overline{w}(s) > \overline{w}(s) - \overline{w}(t) > (-M)[\ln(t) - \ln(s)]$$

for $r_0 \leq s \leq t$. Letting $t \to +\infty$ in last inequality, we obtain a contradiction.

In the case $N > p$ we derive a contradiction as follows. Let $0 < R < r$ and integrate (18) from R to r. This gives

$$r^{N-1}\Phi_p(\overline{w_r})(r) - R^{N-1}\Phi_p(\overline{w_r})(R) = -\int_R^r s^{N-1}\overline{w}^m(s)\mathrm{d}s. \tag{20}$$

Since $\overline{w}$ is continuous at 0, it follows that $\lim_{R\to 0} R^{N-1}\overline{w}_r^{p-1}(R) = L$ exists. Note that L must in fact be zero. Indeed , if $L \neq 0$, then $\overline{w_r} \leq 0$ would not be integrable near $r = 0$ and then $\overline{w}$ would not be continuous at $r = 0$. Thus, letting $R \to 0$ in (20) gives

$$\Phi_p(\overline{w}_r)(r) = -r^{1-N} \int_0^r s^{N-1}\overline{w}^m(s)\mathrm{d}s, \tag{21}$$

for all $r > 0$. Using (17) and (21), one can easily check that $\overline{w} \in C^1[0,\infty)$ with $\overline{w_r}(0) = 0$. From Lemma 1, since

$$N/p < (m+1)/(m-p+1),$$

there is note positive C^1 function on $[0,\infty)$ with zero derivative at 0 which satisfies (18) for all $r > 0$. Thus $\overline{w}$ cannot exist.

These contradictions show that formula (15) is indeed correct. Thus, there exists $c > 0$ such that, for all $t \in (0,T)$ close enough to T,

$$\frac{u_t(0,t)}{u^m(0,t)} \geq c > 0.$$

This can be rewritten as

$$(u^{1-m}(0,t))_t \leq -(m-1)c. \tag{22}$$

Since $\lim_{t\to T} u^{1-m}(0,t) = 0$, integrating (22) from t to T yields

$$u^{1-m} \geq c_1(T-t) \tag{23}$$

for t close to T. Finally, hypotheses (b) and (c) in the theorem, along with formula (23), show that

$$u(x,t) \leq C(T-t)^{-1/(m-1)}$$

for all $(x,t) \in \Gamma$. This completes the proof of the theorem.

Finally, we give lower bounds for the blow-up rates.

Teorem 2 *Assume that the conditions (a)-(f) in Theorem 1 hold. Then there are positive constants c_2, δ such that*

$$u(0,t) \geq c_2(T-t)^{-1/(m-1)}$$

for $t \in (\delta, T)$.

Proof We consider equation (1)(Eq.(1)' being similar). From (1) and consider (c), we get

$$(p-1)(-u')^{p-2}u'' + (N-1)/r|u'|^{p-2}u' + u^m - u^{p-1} = u_t. \tag{24}$$

Since $u'' \leq 0$ at $r = 0$ with $t \in (0,T)$, we see from (24) that

$$u_t(0,t) \leq u^m(0,t) - u^{p-1}(0,t),$$

hence

$$\frac{u_t(0,t)}{u^m(0,t)} \leq 1 - \frac{1}{u^{m-p+1}} \leq 1. \tag{25}$$

Integrating (25) over $(t,s) \subset (\delta, T)$ and leting $s \to T$, we get by condition (f) of Theorem 1

$$u(0,t) \geq c_2(T-t)^{-1/(m-1)}.$$

Remark 1 Combining Theorem 1 and Theorem 2, we conclude that the blow-up rates of radial positive solutions of the (1) or (1)' under the conditions of the theorems are

$$u(0,t) = O((T-t)^{-1/(m-1)}),$$

as t tends to T.

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一类非牛顿渗流方程爆破界的估计

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摘 要: 本文利用拟线性常微分方程解的非存在性定理得到了一类拟线性反应扩散方程(非牛顿渗流方程) 爆破界的估计，从而推广了半线性反应扩散方程 (牛顿渗流方程) 相应结果.