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Systems of Matrix Equations over a Central Algebra *

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Abstract: Let Ω be a finite dimensional central algebra with an involutorial anti-automorphism and chart$\Omega \neq 2$. Two systems of matrix equations over Ω are considered. Necessary and sufficient conditions for the existences of general solutions, and per(skew)selfconjugate solutions of the systems are given, respectively.

Key words: central algebra; system of matrix equations; per(skew)selfconjugate matrix; regular matrix quadruple.

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1. Introduction

Throughout this paper, we denote a finite dimensional central algebra with an involution σ [1, p.112, Definition 1] over a field F by Ω and chart$\Omega \neq 2$, the set of all $m \times n$ matrices over Ω by $\Omega^{m\times n}$, the set of all $n \times n$ invertible matrices over Ω by $GL_n(\Omega)$, the set of all $m \times n$ matrices over $\Omega[\lambda]$ by $\Omega^{m\times n}[\lambda]$, and an $i \times i$ identity matrix by I_i.

Let $A = (a_{ij}) \in \Omega^{m\times n}$, $A^* = (\sigma(a_{m-j+1,n-i+1})) \in \Omega^{n\times m}$. Then $A \in \Omega^{n\times n}$ is called per(skew)selfconjugate if $A = A^*(-A^*)$. The set of all per(skew)selfconjugate matrices is denoted by $\mathbf{C}_n\mathbf{S}_n$).

For matrices A, B over Ω, it is easy to verify that $(A^*)^* = A$, $(AB)^* = B^*A^*$. Define $(A^*)^{-1} = A^{-*}$ if A is invertible. Suppose $A, B \in \Omega^{n\times n}$, $C, D \in \Omega^{m\times m}$, then (A, B, C, D) is called a regular matrix quadruple if there exists $\lambda \in F$ such that $A + \lambda B, C + \lambda D$ are invertible.

Many problems in systems and control theory require the solution of Sylvester's matrix equation $AX - XB = C$ or its generalization $AX - YB = C$. W. E. Roth [2] gave necessary and sufficient conditions for the consistency of the two matrix equations. The matrix equation $AXB - CXD = E$ appears in the numerical solution of implicit ordinary differential equations [3].

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In this paper, the following systems of matrix equations over Ω

$$\left\{\begin{array}{l} A_1X^* - XB_1 = C_1 \\ A_2X^* - XB_2 = C_2 \end{array}\right., \tag{1.1}$$

$$\left\{\begin{array}{l} A_1XB_1 - C_1XD_1 = E_1 \\ A_2XB_2 - C_2XD_2 = E_2 \end{array}\right., \tag{2.2}$$

are considered. Necessary and sufficient conditions are given for the existences of general solutions to (1.1), and a per(skew)selfconjugate solution to (1.2). As a particular case, auxiliary results dealing with the system of Sylvester equations over Ω are also presented.

2. Main results

To begin with the following

Theorem 2.1 *Let $A \in \Omega^{m\times n}[\lambda], B \in \Omega^{n\times m}[\lambda], C \in \Omega^{m\times m}[\lambda]$. Then the matrix equation*

$$AX^* - XB = C \tag{2.1}$$

has a solution X over Ω if and only if there exist $Q \in GL_{n+m}(\Omega)$ such that

$$\begin{bmatrix} -A & -C \\ O & B \end{bmatrix} = Q^* \begin{bmatrix} -A & O \\ O & B \end{bmatrix} Q. \tag{2.2}$$

Proof Suppose that

$$M_0 = \begin{bmatrix} A & O \\ O & B \end{bmatrix}, M_c = \begin{bmatrix} A & C \\ O & B \end{bmatrix}, J = \begin{bmatrix} -I_m & O \\ O & I_n \end{bmatrix}.$$

Let the matrix (2.1) have a solution X over Ω and

$$Q = \begin{bmatrix} I_n & X^* \\ O & I_m \end{bmatrix}, S = \begin{bmatrix} I_m & X \\ O & I_n \end{bmatrix}.$$

Then it is easy to verify that

$$M_0Q = SM_c, Q^*JS = J. \tag{2.3}$$

Hence

$$Q^*JM_0Q = JM_c, \tag{2.4}$$

i.e., (2.2) holds.

Conversely, let (2.2) hold, i.e., (2.4) hold, and $S = J^{-1}Q^{-*}J$. Then (2.3) holds. Suppose that

$$U = \begin{array}{cc} \begin{bmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{bmatrix} & \begin{array}{c} n \\ m \end{array} \\ \begin{array}{cc} n & m \end{array} & \end{array} \in \Omega^{(m+n)\times(m+n)}, V = \begin{array}{cc} \begin{bmatrix} V_1 & V_{12} \\ V_{21} & V_2 \end{bmatrix} & \begin{array}{c} m \\ n \end{array} \\ \begin{array}{cc} m & n \end{array} & \end{array} \in \Omega^{(m+n)\times(m+n)}, \tag{2.5}$$

then

$$U^* = \begin{array}{cc} \begin{bmatrix} U_2^* & U_{12}^* \\ U_{21}^* & U_1^* \end{bmatrix} & \begin{array}{c} m \\ n \end{array} \\ \begin{array}{cc} m & n \end{array} & \end{array} \in \Omega^{(m+n)\times(m+n)}, V^* = \begin{array}{cc} \begin{bmatrix} V_2^* & V_{12}^* \\ V_{21}^* & V_1^* \end{bmatrix} & \begin{array}{c} n \\ m \end{array} \\ \begin{array}{cc} n & m \end{array} & \end{array} \in \Omega^{(m+n)\times(m+n)}.$$

Let

$$T_c = \{(U,V) | M_0 U = V M_c, JU^*JM_0 = M_c J^* V^* J^*\}. \tag{2.6}$$

Then for $(U,V) \in T_c$, we have the following

$$\begin{cases} AU_1 - V_1 A = 0 \\ BU_{21} - V_{21}A = 0 \\ AU_{12} - V_1 C - V_{12}B = 0 \\ BU_2 - V_{21}C - V_2 B = 0 \\ AV_2^* - U_2^* A = CV_{21}^* \\ AV_{12}^* - U_{12}^* B = CV_1^* \\ U_{21}^* A - BV_{21}^* = 0 \\ U_1^* B - BV_1^* = 0 \end{cases}. \tag{2.7}$$

It is easy to verify that T_c is a finite dimensional linear space over F with scalar multiplication and addition defined as follows:

$$(U,V)b = (Ub, Vb), b \in F, (U_1, V_1) + (U_2, V_2) = (U_1 + U_2, V_1 + V_2).$$

For $C = 0$, let T_0 be defined by (2.6). According to (2.3), it is not difficult to verify that

$$(U,V) \in T_c \Leftrightarrow (UQ^{-1}, VS^{-1}) \in T_0.$$

So,

$$\dim T_c = \dim T_0. \tag{2.8}$$

Define a linear map $f : \Omega^{(m+n)\times(m+n)} \times \Omega^{(m+n)\times(m+n)} \longrightarrow \Omega^{(m+n)\times m}$ by the following

$$f(U,V) = \begin{bmatrix} V_1 \\ V_{21} \end{bmatrix}.$$

It is obvious that in the case $C = 0$ we have $(U,V) = (I_{m+n}, I_{m+n}) \in T_0$ and therefore

$$\begin{bmatrix} I_m \\ 0 \end{bmatrix} \in f(T_0). \tag{2.9}$$

Let $f_c = f|_{T_c}, f_0 = f|_{T_0}$. Then it follows from (2.7) that

$$\ker f_c = \ker f_0. \tag{2.10}$$

Suppose U and V are as in (2.5) and

$$K = \begin{bmatrix} U_1 & 0 \\ U_{21} & 0 \end{bmatrix}, L = \begin{bmatrix} V_1 & 0 \\ V_{21} & 0 \end{bmatrix},$$

then it follows from (2.7) that if $(U,V) \in T_c$, then $(K,L) \in T_0$. Hence we have

$$\mathrm{Im} f_c \subseteq \mathrm{Im} f_0. \tag{2.11}$$

It follows from (2.8), (2.10) and

$$\dim\ker f_c + \dim\mathrm{Im} f_c = \dim T_c, \quad \dim\ker f_0 + \dim\mathrm{Im} f_0 = \dim T_0$$

that $\dim\mathrm{Im} f_c = \dim\mathrm{Im} f_0$. Therefor (2.11) yields $\mathrm{Im} f_c = \mathrm{Im} f_0$, i.e. $f(T_c) = f(T_0)$. By (2.9), $\begin{bmatrix} I_m \\ 0 \end{bmatrix} \in f(T_c)$. Consequently there exists $(U,V) \in T_c$ such that $V_1 = I_m$. In view of (2.7), we have

$$AU_{12} - V_{12}B = C, \tag{2.12}$$

$$AV_{12}^* - U_{12}^*B = C. \tag{2.13}$$

Let

$$X = \frac{1}{2}(V_{12} + U_{12}^*). \tag{2.14}$$

Then X is a solution over Ω of the matrix equation (2.1).

Corollary 2.2 *Let $A \in \Omega^{m\times n}, B \in \Omega^{n\times m}, C \in \Omega^{m\times m}$. Then the matrix equation (2.1) is consistent if and only if there exist $Q \in \mathrm{GL}_{n+m}(\Omega)$ such that (2.2) holds.*

Theorem 2.3 *Let $A, B, C \in \Omega^{m\times m}[\lambda]$. Then the Sylvester matrix equation*

$$AX - XB = C \tag{2.15}$$

has a solution $X \in \mathbf{C}_m$ if and only if there exists $Q \in \mathrm{GL}_{2m}(\Omega)$ such that

$$\begin{bmatrix} A & C \\ O & B \end{bmatrix} = Q^{-1}\begin{bmatrix} A & O \\ O & B \end{bmatrix} Q, Q^*\begin{bmatrix} -I_m & O \\ O & I_m \end{bmatrix} Q = \begin{bmatrix} -I_m & O \\ O & I_m \end{bmatrix}.$$

Proof In the proof of Theorem 2.1, let $m = n, V = U, S = Q$ and replace (2.14) by $X = \frac{1}{2}(U_{12} + U_{12}^*)$.Then we can complete the proof by adjusting slightly the rest of the proof of Theorem 2.1.

Theorem 2.4 *Let $A, B, C \in \Omega^{m\times m}[\lambda]$.Then the Sylvester matrix equation (2.15) has a solution $X \in \mathbf{S}_m$ if and only if there exists $Q \in \mathrm{GL}_{2m}(\Omega)$ such that*

$$\begin{bmatrix} A & C \\ O & B \end{bmatrix} = Q^{-1}\begin{bmatrix} A & O \\ O & B \end{bmatrix} Q, Q^*Q = I_{2m}.$$

Proof In the proof of theorem 2.1, let

$$T_c = \{(U,V) | M_0 U = U M_c, U^* M_0 = M_c U^*\}.$$

Then for (2.7),(2.12),(2.13) and (2.14) become respectively the following

$$\begin{cases} AU_1 - U_1A = 0 \\ BU_{21} - U_{21}A = 0 \\ AU_{12} - U_1C - U_{12}B = 0 \\ BU_2 - U_{21}C - U_2B = 0 \\ AU_2^* - U_2^*A + CU_{21}^* = 0 \\ AU_{12}^* - U_{12}^*B + CU_1^* = 0 \\ U_{21}^*A - BU_{21}^* = 0 \\ U_1^*B - BU_1^* = 0 \end{cases},$$

$$AU_{12} - U_{12}B = C, \quad A(-U_{12}^*) - (-U_{12}^*B) = C, \quad X = \frac{1}{2}(U_{12} - U_{12}^*).$$

By adjusting slightly the rest of the proof of Theorem 2.1, we can complete the proof. □

Theorem 2.5 *Let $A_i \in \Omega^{m\times n}, B_i \in \Omega^{n\times m}, C_i \in \Omega^{m\times m}, i = 1,2$. Then the matrix equation (1.1) is consistent if and only if there exist $Q \in \mathrm{GL}_{n+m}(\Omega)$ such that*

$$Q^*\left[\begin{pmatrix} -A_1 & O \\ O & B_1 \end{pmatrix} - \lambda \begin{pmatrix} -A_2 & O \\ O & B_2 \end{pmatrix}\right] Q = \begin{pmatrix} -A_1 & -C_1 \\ O & B_1 \end{pmatrix} - \lambda \begin{pmatrix} -A_2 & -C_2 \\ O & B_2 \end{pmatrix}.$$

Proof The system (1.1) is equivalent to the following

$$(A_1 - \lambda A_2)X^* - X(B_1 - \lambda B_2) = C_1 - \lambda C_2.$$

By Theorem 2.1, we can complete immediately the proof. □

Now we consider the system (1.2) where $A_i, C_i \in \Omega^{m\times m}, B_i, D_i \in \Omega^{n\times n}, E_i \in \Omega^{m\times n}$, and (A_i, C_i, B_i, D_i) is a regular matrix quadruple, $i = 1,2$. There exists $\lambda_i \in F$ such that $C_i + \lambda_i A_i$ and $B_i + \lambda_i D_i$ are invertible, $i = 1,2$. Let

$$\widetilde{A_i} = (C_i + \lambda_i A_i)^{-1}A_i, \widetilde{D_i} = D_i(B_i + \lambda_i D_i)^{-1}, \widetilde{E_i} = (C_i + \lambda_i A_i)^{-1}E_i(B_i + \lambda_i D_i)^{-1}. \quad (2.16)$$

It is clear that the system (1.2) is equivalent to the following

$$(\widetilde{A_1} - \lambda\widetilde{A_2})X - (\widetilde{D_1} - \lambda\widetilde{D_2}) = \widetilde{E_1} - \lambda\widetilde{E_2}.$$

Hence by Theorem 2.3 and Theorem 2.4 we have respectively the following theorems.

Theorem 2.6 *Let $A_i, C_i, B_i, D_i, E_i \in \Omega^{n\times n}$, and (A_i, C_i, B_i, D_i) be a regular matrix quadruple, and $\widetilde{A_i}, \widetilde{D_i}, \widetilde{E_i}$ be defined as (2.16) $i = 1,2$. Then the system (1.2) has a solution $X \in \mathbf{C}_n$ if and only if there exists $Q \in \mathrm{GL}_{2n}(\Omega)$ such that*

$$Q^*\begin{bmatrix} -I_n & O \\ O & I_n \end{bmatrix} Q = \begin{bmatrix} -I_n & O \\ O & I_n \end{bmatrix},$$

$$\begin{bmatrix} \widetilde{A_1} & \widetilde{E_1} \\ O & \widetilde{D_1} \end{bmatrix} - \lambda \begin{bmatrix} \widetilde{A_2} & \widetilde{E_2} \\ O & \widetilde{D_2} \end{bmatrix} = Q^{-1}\left[\begin{pmatrix} \widetilde{A_1} & O \\ O & \widetilde{D_1} \end{pmatrix} - \lambda \begin{pmatrix} \widetilde{A_2} & O \\ O & \widetilde{D_2} \end{pmatrix}\right] Q.$$

Theorem 2.7 *Let $A_i, C_i, B_i, D_i, E_i \in \Omega^{n\times n}$, and (A_i, C_i, B_i, D_i) be a regular matrix quadruple, and $\widetilde{A_i}, \widetilde{D_i}, \widetilde{E_i}$ be defined as (2.16), $i = 1,2$. Then the system (1.2) has a solution $X \in \mathrm{S}_n$ if and only if there exists $Q \in \mathrm{GL}_{2n}(\Omega)$ such that*

$$\begin{bmatrix} \widetilde{A_1} & \widetilde{E_1} \\ O & \widetilde{D_1} \end{bmatrix} - \lambda \begin{bmatrix} \widetilde{A_2} & \widetilde{E_2} \\ O & \widetilde{D_2} \end{bmatrix} = Q^{-1}\left[\begin{pmatrix} \widetilde{A_1} & O \\ O & \widetilde{D_1} \end{pmatrix} - \lambda \begin{pmatrix} \widetilde{A_2} & O \\ O & \widetilde{D_2} \end{pmatrix}\right] Q, Q^*Q = I_{2n}.$$

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中心代数上的矩阵方程组

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摘 要: 设 Ω 是一个特征非 2 的具有对合反自同构的有限维中心代数．本文研究 Ω 上的两个矩阵方程组，分别给出了其有一般解和次 (斜) 自共轭解的充要条件．