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# A Note on the Integrity of Certain Series *

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**Abstract:** In this paper, using the theory of Pell equation, the authors discuss the integrity of certain series related to generalized Lucas numbers. Under some conditions, the integrity of certain series involving generalized Lucas numbers is completely solved.

**Key words:** Fibonacci numbers; Lucas numbers; series.

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## 1. Introduction

The generalized Fibonacci and Lucas numbers can be defined as follows:

$$U_n^{(d)} = \frac{\alpha^{n+d} - \beta^{n+d}}{\alpha - \beta} \quad \text{and} \quad V_n^{(d)} = \alpha^{n+d} + \beta^{n+d}, \tag{1}$$

where $\alpha = (p+\sqrt{\Delta})/2$, $\beta = (p-\sqrt{\Delta})/2$, $\Delta = p^2 - 4q$, $p$ and $q$ are integers with $p^2 - 4q > 0$, and $d$ is a nonnegative integer. Throughout this paper, we assume that $p > 0$ and $q \neq 0$. It is clear that $U_n^{(0)} = F_n$ and $V_n^{(0)} = L_n$ when $p = -q = 1$, where $\{F_n\}$ and $\{L_n\}$ denote the Fibonacci sequence and the Lucas sequence, respectively. From the definitions of $U_n^{(d)}$ and $V_n^{(d)}$, we know that $\{U_n^{(d)}\}$ and $\{V_n^{(d)}\}$ satisfy the linear recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, \quad (n \geq 2).$$

For convenience, we write $U_n^{(0)} = U_n$ and $V_n^{(0)} = V_n$.

Recently, many authors have investigated the integrity of the following series (see [1–5]):

$$S_U(x;p,q) = \sum_{n=0}^{\infty} \frac{U_n}{x^n}, \quad S_V(x;p,q) = \sum_{n=0}^{\infty} \frac{V_n}{x^n}, \quad \text{and} \quad T_k(x) = \sum_{n=0}^{\infty} \frac{V_{kn}}{x^n}, \quad k > 0,$$

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where $q = \pm 1$.

In this paper, we discuss the integrity of the infinite series $R_V^{(d)}(x) = \sum_{n=0}^{\infty} \frac{V_n^{(d)}}{x^n}$, $d \geq 1$. The integrities of $R_V^{(1)}(x)$ $(q = \pm 1)$ and $R_V^{(2)}(x)$ $(q = 1)$ will be completely solved.

In what follows, we will make use of the following identity:

$$V_n + pU_n = 2U_{n+1}. \tag{2}$$

## 2. Main results

In this section, we suppose that $q = \pm 1$.

We state the main results of this paper.

**Theorem 1** *For $q = \pm 1$, we have:*

*(I) If $q = -1$, the positive rational values of $x$ for which $R_V^{(1)}(x)$ is integral are given by*

$$x = \frac{U_{2m+1} + 1}{U_{2m} - p} \quad (m \geq 2).$$

*The corresponding values of $R_V^{(1)}(x)$ are given by*

$$R_V^{(1)}\left(\frac{U_{2m+1} + 1}{U_{2m} - p}\right) = U_{2m}, \quad (m \geq 2).$$

*(II) If $q = 1$ and $p > 2$, the positive rational values of $x$ for which $R_V^{(1)}(x)$ is integral are given by*

$$x = \frac{U_{m+1} - 1}{U_m - p}, \quad (m \geq 3).$$

*The corresponding values of $R_V^{(1)}(x)$ are given by*

$$R_V^{(1)}\left(\frac{U_{m+1} - 1}{U_m - p}\right) = U_m, \quad (m \geq 3).$$

**Proof** By (1) and the geometric series formula, we have

$$R_V^{(d)}(x) = \frac{x(V_d x - qV_{d-1})}{x^2 - px + q}, \quad (|x| > \max(|\alpha|, |\beta|)).$$

Let $R_V^{(d)}(x) = \frac{x(V_d x - qV_{d-1})}{x^2 - px + q} = r$, where $r$ is an integer. Then we obtain the second–degree equation

$$(r - V_d)x^2 - (pr - qV_{d-1})x + qr = 0. \tag{3}$$

The roots of (3) are $x = \frac{pr - qV_{d-1} \pm \sqrt{D}}{2(r - V_d)}$, where $D = (pr - qV_{d-1})^2 - 4qr(r - V_d)$.

When $x$ is rational, the equation

$$D = y^2 \tag{4}$$

must hold, where $y$ is an integer.

When $d=1$, $V_d=p$ and $V_{d-1}=2$. Thus $D=\Delta r^2+4$, where $\Delta=p^2\pm 4$. It follows from the theory of Pell equation (see [6]) and $r-p\neq 0$ that the solutions of (4) are given by $y=V_{2m}, r=U_{2m}(m\geq 2)$, if $q=-1$, and $y=V_m, r=U_m(m\geq 2)$, if $q=1$. Hence, the roots of (3) are

$$x=\begin{cases}\dfrac{pU_{2m}+2\pm V_{2m}}{2(U_{2m}-p)}, & (m\geq 2), \quad \text{if} \quad q=-1,\\ \dfrac{pU_m-2\pm V_m}{2(U_m-p)}, & (m\geq 3), \quad \text{if} \quad q=1.\end{cases}$$

We require only the positive root of (3), which is shown to be

$$x=\begin{cases}\dfrac{pU_{2m}+2+V_{2m}}{2(U_{2m}-p)}, & (m\geq 2), \quad \text{if} \quad q=-1,\\ \dfrac{pU_m-2+V_m}{2(U_m-p)}, & (m\geq 3), \quad \text{if} \quad q=1.\end{cases}$$

Owing to (2), we obtain

$$x=\begin{cases}\dfrac{U_{2m+1}+1}{U_{2m}-p}, & (m\geq 2), \quad \text{if} \quad q=-1,\\ \dfrac{U_{m+1}-1}{U_m-p}, & (m\geq 3), \quad \text{if} \quad q=1.\end{cases}$$

From (1), we can verify that

$$\frac{U_{2m+1}+1}{U_{2m}-p}>\alpha \quad (q=-1, \quad m\geq 2) \quad \text{and} \quad \frac{U_{m+1}-1}{U_m-p}>\alpha \quad (q=1, \quad m\geq 3). \quad \square$$

**Theorem 2** *If $q=1$ and $p>2$, the positive rational values of $x$ for which $R_V^{(2)}(x)$ is integral are given by*

$$x=\frac{U_{m+1}-p}{U_m-p^2+1}, \quad (m\geq 4).$$

*The corresponding values of $R_V^{(2)}(x)$ are given by*

$$R_V^{(2)}\left(\frac{U_{m+1}-p}{U_m-p^2+1}\right)=U_m-1, \quad (m\geq 4).$$

**Proof** The proof of Theorem 2 is similar to that of (I) of Theorem 1 except $V_2=p^2-2$ and $D=\Delta(r+1)^2+4$ and omitted here. $\square$

**Remark** Consider the series $R_k^{(dk)}(x)=\sum_{n=0}^{\infty}\frac{V_{nk}^{(dk)}}{x^n}, k>0$. Clearly, $R_1^{(d)}(x)=R_V^{(d)}(x)$. For $R_k^{(k)}(x)$ and $R_k^{(2k)}(x)$, we have the following conclusions:

**Theorem 1′** *If $q=-1$ and $k\geq 0$, the positive rational values of $x$ for which $R_{2k+1}^{(2k+1)}(x)$ is integral are given by*

$$x=\frac{U_{(2m+1)(2k+1)}+U_{2k+1}}{U_{2m(2k+1)}-U_{4k+2}}, \quad (m\geq 2).$$

*The corresponding values of* $R_{2k+1}^{(2k+1)}(x)$ *are given by*

$$R_{2k+1}^{(2k+1)}\left(\frac{U_{(2m+1)(2k+1)}+U_{2k+1}}{U_{2m(2k+1)}-U_{4k+2}}\right)=\frac{U_{2m(2k+1)}}{U_{2k+1}},\quad (m\geq 2).$$

**Theorem 2′** *If* $q=-1$ *and* $k\geq 1$, *the positive rational values of* $x$ *for which* $R_{2k}^{(2k)}(x)$ *is integral are given by*

$$x=\frac{U_{2k+2mk}-U_{2k}}{U_{2mk}-U_{4k}},\quad (m\geq 3).$$

*The corresponding values of* $R_{2k}^{(2k)}(x)$ *are given by*

$$R_{2k}^{(2k)}\left(\frac{U_{2k+2mk}-U_{2k}}{U_{2mk}-U_{4k}}\right)=\frac{U_{2mk}}{U_{2k}},\quad (m\geq 3).$$

**Theorem 3′** *If* $q=1$, $p>2$, *and* $k\geq 1$, *the positive rational values of* $x$ *for which* $R_{k}^{(k)}(x)$ *is integral are given by*

$$x=\frac{U_{(m+1)k}-U_{k}}{U_{mk}-U_{2k}},\quad (m\geq 3).$$

*The corresponding values of* $R_{k}^{(k)}(x)$ *are given by*

$$R_{k}^{(k)}\left(\frac{U_{(m+1)k}-U_{k}}{U_{mk}-U_{2k}}\right)=\frac{U_{mk}}{U_{k}},\quad (m\geq 3).$$

**Theorem 4′** *If* $q=-1$ *and* $k\geq 1$, *the positive rational values of* $x$ *for which* $R_{2k}^{(4k)}(x)$ *is integral are given by*

$$x=\frac{U_{2k(m+1)}-U_{4k}}{U_{2mk}-V_{2k}^{2}U_{2k}+U_{2k}},\quad (m\geq 4).$$

*The corresponding values of* $R_{2k}^{(4k)}(x)$ *are given by*

$$R_{2k}^{(4k)}\left(\frac{U_{2k(m+1)}-U_{4k}}{U_{2mk}-V_{2k}^{2}U_{2k}+U_{2k}}\right)=\frac{U_{2mk}}{U_{2k}}-1,\quad (m\geq 4).$$

**Theorem 5′** *If* $q=1$, $p>2$, *and* $k\geq 2$, *the positive rational values of* $x$ *for which* $R_{k}^{(2k)}(x)$ *is integral are given by*

$$x=\frac{U_{(m+1)k}-U_{2k}}{U_{mk}-V_{k}^{2}U_{k}+U_{k}},\quad (m\geq 4).$$

*The corresponding values of* $R_{k}^{(2k)}(x)$ *are given by*

$$R_{k}^{(2k)}\left(\frac{U_{(m+1)k}-U_{2k}}{U_{mk}-V_{k}^{2}U_{k}+U_{k}}\right)=\frac{U_{mk}}{U_{k}}-1,\quad (m\geq 4).$$

**Proof** Let

$$V_{n}^{\prime(d)}=\alpha^{nk+dk}+\beta^{nk+dk}=V_{nk}^{(dk)}\quad\text{and}\quad U_{n}^{\prime(d)}=\frac{\alpha^{nk+dk}-\beta^{nk+dk}}{\alpha^{k}-\beta^{k}}=\frac{U_{nk}^{(dk)}}{U_{k}}.\qquad(5)$$

In the meantime, during the proofs of Theorems 1′–5′, we will use the identity

$$U_{2t} = U_t V_t. \tag{6}$$

In fact, the sequences $\{U_n'^{(d)}\}$ and $\{V_n'^{(d)}\}$ satisfy the linear recurrence relation

$$W_n = V_k W_{n-1} - q^k W_{n-2},\ \ n \geq 2.$$

From (5) and applying Theorems 1–2 to the sequences $\{V_n'^{(1)}\}$ and $\{V_n'^{(2)}\}$, and noticing (6), we can obtain the conclusions of Theorems 1′–5′.

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# 关于一些级数取整值问题的注记


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**摘 要**: 利用 Pell 方程的理论，讨论了与广义 Lucas 数有关的一些级数的取整值问题．在一定条件下，解决了一些级数的取整值问题．