Journal of Mathematical Research & Exposition
Vol.23, No.1, 33-39, February, 2003

Ishikawa Iteration Process for Approximating Fixed Points of Nonexpansive Mappings *

ZENG Lu-chuan
(Dept. of Math., Shanghai Normal University, Shanghai 200234, China)

Abstract: Let$E$ be a uniformly convex Banach space which satisfies Opial's condition or has a Frechet differentiable norm, and C be a bounded closed convex subset of E. If $T: C \to C$ is a nonexpansive mapping, then for any initial data $x_0 \in C$, the Ishikawa iteration process $\{x_n\}$, defined by $x_n = t_n T(s_n T x_n + (1-s_n)x_n) + (1-t_n)x_n, n \geq 0$, converges weakly to a fixed point of T, where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0,1]$ with some restrictions.

Key words: fixed point; nonexpansive mapping; Ishikawa iteration process; uniformly convex Banach space.

Classification: AMS(2000) 47H10,47H17/CLC number: O177.91

Document code: A **Article ID**: 1000-341X(2003)01-0033-07

1. Introduction and preliminaries

Let C be a nonempty subset of a Banach space$E$. Then a mapping$T: C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It has been shown that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E then every nonexpansive mapping $T: C \to C$ has a fixed point; see [1,6] for more details. In 1974, Ishikawa[5] introduced a new iteration procedure for approximating fixed points of pseudo-contractive compact mappings as follows:

$$x_{n+1} = t_n T(s_n T x_n + (1-s_n)x_n) + (1-t_n)x_n, \quad n = 0, 1, 2, \cdots, \tag{I}$$

where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0,1]$ satisfying certain restrictions.

Recently, Tan and Xu[3] proved the following interesting result which generalizes Theorem 2 of Reich[8]:

Theorem 1.1[3,Theorem1] *Let E be a uniformly convex Banach space that satisfies the*

*Received date: 2000-04-03
Foundation item: Supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China.
Biography: ZENG Lu-chuan (1965-), male, Ph.D., Professor.

Opial condition or whose norm is Frechet differentiable, C be a bounded closed convex subset of E, and $T : C \to C$ be a nonexpansive mapping. Then for any initial guess x_0 in C, the Ishikawa iteration process $\{x_n\}$ defined by (I), with the restrictions that $\overline{\lim}_{n\to\infty} s_n < 1$, $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty$, and $\sum_{n=0}^{\infty} s_n(1-t_n) < \infty$, converges weakly to a fixed point of T.

Furthermore, Zeng[9] proved the following weak convergence theorem which complements Theorem 1 of Tan and Xu[3].

Theorem 1.2[9,Theorem1] *Let E be a uniformly convex Banach space that satisfies the Opial condition or whose norm is Frechet differentiable, C be a bounded closed convex subset of E, and $T : C \to C$ be a nonexpansive mapping. Then for any initial guess x_0 in C, the Ishikawa iteration process $\{x_n\}$ defined by (I), with the restrictions that $\limsup_{n\to\infty} s_n$ is less than 1 and for any subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=0}^{\infty}$, $\sum_{k=0}^{\infty} t_{n_k}(1-t_{n_k})$ diverges, converges weakly to a fixed point of T.*

In this paper, we investigate the problem of approximating fixed points of nonexpansive mappings by the Ishikawa iteration process (I). Motivated and inspired by the recent research work of Tan and Xu[3] and Zeng[9], we show that if E is a uniformly convex Banach space satisfying Opial's condition or with a Frechet differentiable norm, C is a bounded closed convex subset of E, and $T : C \to C$ is a nonexpansive mapping, then for any initial guess $x_0 \in C$, the Ishikawa iteration process $\{x_n\}$, defined by (I), with the restriction that for any subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=0}^{\infty}$, $\sum_{k=0}^{\infty} t_{n_k} s_{n_k}(1-s_{n_k})$ diverges, converges weakly to a fixed point of T. Moreover, we also give the strong convergence theorems of the Ishikawa iteration process (I) for nonexpansive mappings. On one hand, our results complement and improve Theorem 1 of Tan and Xu[3] and Theorem 1 of Zeng[9]. On the other hand, our results also generalize, to a certain extent, Theorem 2 of Reich[8].

Now we give some preliminaries. Let E be a Banach space. Recall that E is said to satisfy Opial's condition [7] if for each sequence $\{x_n\}$ in E the condition $x_n \to x$ weakly implies

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } \ y \in E, \text{ and } y \neq x. \tag{1.1}$$

It is known that (1.1) is equivalent to the analogous condition obtained by replacing $\liminf_{n\to\infty}$ with $\limsup_{n\to\infty}$. Recall also that E is said to have a Frechet differentiable norm if, for each x in $S(E)$, the unit sphere of E, the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists and is attained uniformly in $y \in S(E)$. In this case, we have

$$\frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \le \frac{1}{2}\|x+h\|^2 \le \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + b(\|h\|) \tag{1.2}$$

for all $x, h \in E$, where J is the normalized duality map from E to E^* defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\},$$

$\langle \cdot,\cdot\rangle$is the duality pairing between E and E^*, and b is a function defined on$[0,\infty)$ such that $\lim_{t\downarrow 0} b(t)/t = 0$.

Throughout this paper, we suppose that C is a bounded closed convex subset of a uniformly convex Banach space E, $T : C \to C$ is a nonexpansive mapping, and $\{x_n\}$is the

Ishikawa iterates defined by (I). Denote by $F(T)$ the set of fixed points of T.

Lemma 1.1[3] *Suppose that* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are two sequences of nonnegative numbers satisfying the inequality:*

$$a_{n+1} \leq a_n + b_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 1.2[3] *The following (1)-(2) are valid:*

(1) For each $f \in F(T)$, $\{\|x_n - f\|\}$ *is nonincreasing and the limit* $\lim_{n\to\infty} \|x_n - f\|$ *exists;*

(2) Suppose, in addition, that E *has a Frechet differentiable norm. Then for every* f_1, f_2 *in* $F(T)$ *and* $0 < t < 1$, *the limit*

$$\lim_{n\to\infty} \|tx_n + (1-t) f_1 - f_2\|$$

exists.

Denote by Γ the set of all strictly increasing convex functions $\gamma : R^+ \to R^+$ with $\gamma(0) = 0$. Let $C \subseteq E$. Mapping $T : C \to C$ is said to be of type (γ) iff

$$\bigvee_{\gamma\in\Gamma} \bigwedge_{x,y\in C} \bigwedge_{0\leq c\leq 1} \gamma(\|cTx + (1-c)Ty - T(cx + (1-c)y)\|) \leq \|x - y\| - \|Tx - Ty\|$$

In 1991, Gornicki[4] gave the following lemma.

Lemma 1.3 (cf. Gornicki[4,Lemma3.1]) *Let* C *be a bounded closed convex subset of a B-convex Banach space* E. *Assume that* $S : C \to C$ *is a nonexpansive mapping of type* (γ). *If* $y_n \to y$ *weakly* $(y_n, y \in C, n = 1, 2, \cdots)$, *then there is a function* $g \in \Gamma$ *such that*

$$g(\|y - Sy\|) \leq \liminf_{n\to\infty} \|y_n - Sy_n\|.$$

Remark 1.1 If E is a uniformly convex Banach space, then each nonexpansive mapping $T : C \to C$ is of type (γ), and γ can be chosen to depend only on $\operatorname{diam}(C)$ and not on T[2].

More suitable to our purpose is a variant of Lemma 1.3 for nonexpansive map-pings in a uniformly convex Banach space.

Corollary 1.1 *Let* C *be a bounded closed convex subset of a uniformly convex Banach space* E. *Suppose that* $S : C \to C$ *is a nonexpansive mapping. If* $y_n \to y$ *weakly* $(y_n, y \in C, n = 1, 2, \cdots)$, *then there exists a strictly increasing convex function* $g : R^+ \to R^+$ *with* $g(0) = 0$ *such that*

$$g(\|y - Sy\|) \leq \liminf_{n\to\infty} \|y_n - Sy_n\|.$$

We remind the reader of the following fact:

$$[g : R^+ \to R^+, g(0) = 0, g \text{ strictly increasing convex}] \Rightarrow [g \text{ is continuous}].$$

2. Main results

Theorem 2.1 *Let E be a uniformly convex Banach space which satisfies Opial's condition or has a Frechet differentiable norm, C be a bounded closed convex subset of E, and $T : C \to C$ be a nonexpansive mapping. Then for any initial guess $x_0$in C, the Ishikawa iteration process $\{x_n\}$, defined by (I), with the restriction that for any subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=0}^{\infty}$, $\sum_{k=0}^{\infty} t_{n_k} s_{n_k} (1 - s_{n_k})$ diverges, converges weakly to a fixed point of T.*

Proof Set $y_n = s_n T x_n + (1 - s_n) x_n$. Then

$$x_{n+1} = t_n T y_n + (1 - t_n) x_n.$$

Let f be in $F(T)$. It follows from Lemma 1.2 (1) that the limit $\lim_{n\to\infty} \|x_n - f\|$ exists. We may assume $\lim_{n\to\infty} \|x_n - f\| \neq 0$.

Firstly, we show that if $\{n_k\}_{k=0}^{\infty}$ is a subsequence of $\{n\}_{n=0}^{\infty}$ such that $\sum_{k=0}^{\infty} t_{n_k} s_{n_k} (1 - s_{n_k})$ diverges, then $\liminf_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$. Indeed, by the method of Tan and Xu[3], we obtain

$$\begin{aligned} \|y_n - f\| &= \|s_n (Tx_n - f) + (1 - s_n)(x_n - f)\| \\ &\leq \|x_n - f\| \left[1 - 2s_n (1 - s_n) \delta_E (\|Tx_n - x_n\| / \|x_n - f\|)\right], \end{aligned} \tag{2.1}$$

where δ_E is the modulus of convexity of E defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \text{ and } \|x - y\| \geq \varepsilon\right\}$$

for $0 \leq \varepsilon \leq 2$. Since

$$\begin{aligned} \|x_{n+1} - f\| &= \|t_n (Ty_n - f) + (1 - t_n)(x_n - f)\| \\ &\leq t_n \|y_n - f\| + (1 - t_n) \|x_n - f\|, \end{aligned}$$

we have from (2.1)

$$\begin{aligned} \|x_{n+1} - f\| &\leq t_n \|x_n - f\| \left[1 - 2s_n (1 - s_n) \delta_E (\|Tx_n - x_n\| / \|x_n - f\|)\right] + (1 - t_n) \|x_n - f\| \\ &= \|x_n - f\| \left[1 - 2t_n s_n (1 - s_n) \delta_E (\|Tx_n - x_n\| / \|x_n - f\|)\right]. \end{aligned} \tag{2.2}$$

Put

$$D_n = 1 - 2t_n s_n (1 - s_n) \delta_E (\|Tx_n - x_n\| / \|x_n - f\|) \text{ for each } n \geq 0.$$

Then it is clear that $0 \leq D_n \leq 1$ for each $n \geq 0$. From (2.2) it follows that for each $k \geq 0$,

$$\begin{aligned} \|x_{n_{k+1}} - f\| &\leq D_{n_{k+1}-1} \|x_{n_{k+1}-1} - f\| \leq D_{n_{k+1}-1} D_{n_{k+1}-2} \cdots D_{n_k+1} D_{n_k} \|x_{n_k} - f\| \\ &\leq D_{n_k} \|x_{n_k} - f\| \\ &= \|x_{n_k} - f\| \left[1 - 2t_{n_k} s_{n_k} (1 - s_{n_k}) \delta_E (\|Tx_{n_k} - x_{n_k}\| / \|x_{n_k} - f\|)\right]. \end{aligned} \tag{2.3}$$

Now it is readily seen from (2.3) that

$$\sum_{k=0}^{\infty} t_{n_k} s_{n_k} \left(1 - s_{n_k}\right) \delta_E \left(\|Tx_{n_k} - x_{n_k}\| / \|x_{n_k} - f\|\right) < \infty.$$

But, since $\sum_{k=0}^{\infty} t_{n_k} s_{n_k} \left(1 - s_{n_k}\right)$ diverges, we deduce

$$\liminf_{k\to\infty} \delta_E \left(\|Tx_{n_k} - x_{n_k}\| / \|x_{n_k} - f\|\right) = 0.$$

It then follows that

$$\liminf_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0,$$

since δ_E is strictly increasing and continuous and $\lim_{n\to\infty} \|x_n - f\| \neq 0$.

Secondly, we prove that $\{x_n\}$ converges weakly to a fixed point of T.

By $\omega_w(x_n)$ we denote the set of all weak subsequential limits of $\{x_n\}$, i.e., the weak ω-lim set of the sequence$\{x_n\}$,

$$\omega_w(x_n) = \left\{ y \in E : y = \text{weak - } \lim_{k\to\infty} x_{n_k} \text{ for some } n_k \uparrow \infty \right\}.$$

Now, we assert that $\omega_w(x_n) \subseteq F(T)$. Indeed, let $x_{n_k} \to x$ weakly as$k \to \infty$. Then it is readily seen that for subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=0}^{\infty}$, we have $\sum\limits_{k=0}^{\infty} t_{n_k} s_{n_k} \left(1 - s_{n_k}\right) = \infty$. Thus, it follows immediately from the above argument that $\liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$. And also, by Corollary 1.1 there is a strictly increasing continuous convex function $g : R^+ \to R^+$ with $g(0) = 0$ such that

$$g(\|x - Tx\|) \leq \liminf_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

Obviously, we obtain $x = Tx$, which implies $x \in F(T)$. To show that $\{x_n\}$ converges weakly to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. To this end, we first suppose that E satisfies Opial's condition and suppose $p \neq q$ are in $\omega_w(x_n)$. Then p =weak-$\lim\limits_{k\to\infty} x_{n_k}$ and q =weak-$\lim_{j\to\infty} x_{m_j}$ for some subsequences $\{n_k\}$ and $\{m_j\}$. By Lemma 1.2 (1) and Opial's condition of E, we derive

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\ &= \lim_{j\to\infty} \left\|x_{m_j} - q\right\| < \lim_{j\to\infty} \left\|x_{m_j} - p\right\| = \lim_{n\to\infty} \|x_n - p\|, \end{aligned}$$

arriving at a contraction. This proves the theorem in the case when E satisfies Opial's condition. On the other hand, we assume that$E$ has a Frechet differentiable norm. Exploiting inequality (1.2), and Lemma 1.2 (2), we see that for each $f_1, f_2 \in F(T)$, the limit $\lim_{n\to\infty} \langle x_n, J(f_1 - f_2)\rangle$ exists; see Tan and Xu [3, the proof of Theorem 1] for more details. In particular,

$$\langle p - q, J(f_1 - f_2)\rangle = 0 \text{ for all } p, q \in \omega_w(x_n). \tag{2.4}$$

Since $\omega_w(x_n) \subseteq F(T)$, for any $p, q \in \omega_w(x_n)$, by replacing f_1, f_2 in (2.4) with p, q, respectively, we derive

$$\|p-q\|^2 = \langle p-q, J(p-q)\rangle = 0.$$

This shows that $\omega_w(x_n)$ must be a singleton. The proof is complete.

Theorem 2.2 *Let E be a uniformly convex Banach space which satisfies Opial's condition or has a Frechet differentiable norm, C be a bounded closed convex subset of E, and $T: C \to C$ be a nonexpansive mapping. Then for any initial guess x_0 in C, the Ishikawa iteration process $\{x_n\}$, defined by (I), with the restrictions that $\sum_{n=0}^{\infty} t_n s_n (1-s_n)$ diverges, and $\sum_{n=0}^{\infty} s_n (1-t_n)$ converges, converges weakly to a fixed point of T.*

Proof Using (2.2) and the assumption that $\sum_{n=0}^{\infty} t_n s_n (1-s_n)$ diverges, we obtain

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2.5}$$

On the other hand, by the method of Tan and Xu [3], we can show

$$\|Tx_{n+1} - x_{n+1}\| \le [1 + 2s_n(1-t_n)]\,\|Tx_n - x_n\|.$$

Since $\sum_{n=0}^{\infty} s_n (1-t_n)$ converges and $\{\|Tx_n - x_n\|\}$ is bounded, it follows from Lemma 1.1 that $\lim_{n\to\infty}\|Tx_n - x_n\|$ exists and equals zero by (2.5).

The remainder of the proof is seen to be similar to that of Theorem 1 in Tan and Xu[3]. Thus, we omit it.

Further, we give strong convergence theorems of the Ishikawa iteration process (I) for nonexpansive mappings.

Theorem 2.3 *Suppose that E is a uniformly convex Banach space and T, C, and $\{x_n\}$ are as in Theorem 2.1. Suppose also that the range of C under T is contained in a compact subset of E. Then the Ishikawa iterates $\{x_n\}$ converge strongly to a fixed point of T.*

A mapping $T: C \to C$ is said to satisfy Condition A[10] if there is a nondecreasing function$f: R^+ \to R^+$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, such that

$$\|x - Tx\| \ge f(d(x, F(T))) \text{ for all } x \in C,$$

where $d(x, F(T)) = \inf_{z\in F(T)} \|x - z\|$.

Theorem 2.4 *Let E be a uniformly convex Banach space, and let T, C be as in Theorem 2.1. If T satisfies Condition A, then for any initial guess $x_0 \in C$, the Ishikawa iteration process $\{x_n\}$, defined by (I), with the restriction that for any subsequence $\{n_k\}_{k=0}^{\infty}$ of $\{n\}_{n=0}^{\infty}$, $\sum_{k=0}^{\infty} t_{n_k} s_{n_k}(1-s_{n_k})$ diverges, converges strongly to a fixed point of T.*

Remark 2.1 It is obviously seen that our Theorems 2.3, 2.4 also complement and improve Theorems 2, 3 of Tan and Xu[3], respectively, by removing their restriction, $\sum_{n=0}^{\infty} s_n(1-t_n) < \infty$. In addition, according to Theorem 2.2, we may also obtain the results corresponding to Theorems 2.3, 2.4.

References:

[1] BROWDER F E. *Nonlinear operators and nonlinear equations of evolution in Banach spaces* [J]. Proc. Sympos. Pure Math., Vol. **18**, Amer. Math. Soc., Providence, RI, 1976.

[2] BRUCK R E. *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces* [J]. Isreal J. Math., 1979, **32**: 107–116.

[3] TAN K K, XU H K. *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process* [J]. J. Math. Anal. Appl., 1993, **178**: 301–308.

[4] GORNICKI J. *Nonlinear ergodic theorems for asymptotically nonexpansive mappings in Banach spaces satisfying Opial's condition* [J]. J. Math. Anal. Appl., 1991, **161**: 440–446.

[5] ISHIKAWA S. *Fixed points by a new iteration method* [J]. Proc. Amer. Math. Soc., 1974, **44**: 147–150.

[6] KIRK W A. *A fixed point theorem for mappings which do not increase distance* [J]. Amer. Math. Monthly, 1965, **72**: 1004–1006.

[7] OPIAL Z. *Weak convergence of the sequence of successive approximations for nonexpansive mappings* [J]. Bull. Amer. Math. Soc., 1967, **73**: 595–597.

[8] REICH S. *Weak convergence theorems for nonexpansive mappings in Banach spaces* [J]. J. Math. Anal. Appl., 1979, **67**: 274–276.

[9] ZENG L C. *A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process* [J]. J. Math. Anal. Appl., 1998, **226**: 245–250.

[10] SENTER H F, DOTSON W G JR. *Approximating fixed points of nonexpansive mappings* [J]. Proc. Amer. Math. Soc., 1974, **44**: 375–380.

关于非扩张映象的不动点逼近的 Ishikawa 迭代程序

曾 六 川

(上海师范大学数学系，上海 200234)

摘 要: 设 E 是一致凸 Banach 空间，满足 Opial 条件或具有 Frechet 可微范数. 又设 C 是 E 的有界闭凸子集. 若 $T: C \to C$ 是非扩张映象，则对任给的初始数据 $x_0 \in C$，由 Ishikawa 迭代程序

$$x_{n+1} = t_n T\left(s_n T x_n + (1 - s_n) x_n\right) + (1 - t_n) x_n, n \geq 0,$$

定义的序列 $\{x_n\}$ 弱收敛到 T 的不动点，其中，$\{t_n\}, \{s_n\}$ 是区间 $[0,1]$ 中满足某些限制的序列.