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On Generalizations of a Kind of Hilbert's Integral Inequality *

LÜ Zhong-xue[1], *Xie Hong-zheng*[2]

(1. Dept. of Basi. Sci. of Tech. College, Xuzhou Normal University, Jiangsu 221011, China;
2. Dept. of Math., Harbin Institute of Technology, Heilongjiang 150001, China)

Abstract: In this paper, we generalize the equivalent form of Hilbert's integral inequality by introducing parameters p, q, a, b and t.

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1. Introduction

The following inequalities (1) is well-known as the equivalent form of Hilbert's integral inequality

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\mathrm{d}x\right)^2 \mathrm{d}y \le \pi^2 \int_0^\infty f^2(x)\mathrm{d}x, \tag{1.1}$$

where π^2 is the best value(cf.[1,Chap 9]).

Recently,Yang Bicheng[2] gave the following result.

$$\int_a^b \left(\int_a^b \frac{f(x)}{x+y}\mathrm{d}x\right)^2 \mathrm{d}y \le \pi^2 \left(1-\sqrt[4]{\frac{a}{b}}\right)^2 \int_a^b f^2(x)\mathrm{d}x, \tag{1.2}$$

$$\int_0^b \left(\int_0^b \frac{f(x)}{x+y}\mathrm{d}x\right)^2 \mathrm{d}y \le \pi^2 \int_0^b \left(1-\frac{1}{2}\sqrt[2]{\frac{x}{b}}\right) f^2(x)\mathrm{d}x, \tag{1.3}$$

$$\int_a^\infty \left(\int_a^\infty \frac{f(x)}{x+y}\mathrm{d}x\right)^2 \mathrm{d}y \le \pi^2 \int_a^\infty \left(1-\frac{1}{2}\sqrt[2]{\frac{a}{x}}\right) f^2(x)\mathrm{d}x, \tag{1.4}$$

which generalize the inequality of (1.1) by introducing two parameters a and b.

In this paper,some generalizations of (1.1)-(1.4) are given in the following theorems.

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Theorem 1 *Let* $0<a<b, p>1, \frac{1}{p}+\frac{1}{q}=1, f\in L^p[0,\infty)$, *then*

$$\int_a^b\left(\int_a^b\frac{f(x)}{x+y}\mathrm{d}x\right)^p\mathrm{d}y\le[\max\{C,D\}]^p\int_a^b f^p(x)\mathrm{d}x, \tag{1.5}$$

$$\int_0^b\left(\int_0^b\frac{f(x)}{x+y}\mathrm{d}x\right)^p\mathrm{d}y\le B\left(\frac{1}{p},\frac{1}{q}\right)^{p-1}\int_0^b\left(B\left(\frac{1}{p},\frac{1}{q}\right)-\theta_{1/p}\left(\frac{x}{b}\right)^{1/q}\right)f^p(x)\mathrm{d}x, \tag{1.6}$$

$$\int_a^\infty\left(\int_a^\infty\frac{f(x)}{x+y}\mathrm{d}x\right)^p\mathrm{d}y\le B\left(\frac{1}{p},\frac{1}{q}\right)^{p-1}\int_a^\infty\left(B\left(\frac{1}{p},\frac{1}{q}\right)-\theta_{1/q}\left(\frac{a}{x}\right)^{1/p}\right)f^p(x)\mathrm{d}x, \tag{1.7}$$

where

$$C=B(\frac{1}{p},\frac{1}{q})-2[\theta_{1/p}\theta_{1/q}(\frac{a}{b})^{1/p}]^{1/2},\quad D=B(\frac{1}{p},\frac{1}{q})-2[\theta_{1/p}\theta_{1/q}(\frac{a}{b})^{1/q}]^{1/2},$$

$$\theta_s=\int_0^1\frac{1}{1+u}(\frac{1}{u})^s\mathrm{d}u,\quad s=\frac{1}{p},\frac{1}{q},$$

$B(m,n)(m,n>0)$ *is* β*-function, and* $f\ge 0$.

Theorem 2 *Let* $0<a<b, p>1, \frac{1}{p}+\frac{1}{q}=1, 1-\frac{1}{pq}<t\le 1, f\in L^p[0,\infty)$, *then*

$$\int_a^b\left(\int_a^b\frac{f(x)}{(x+y)^t}\mathrm{d}x\right)^p\mathrm{d}y\le\max\{E,F,G,H\}\int_a^b x^{p(1-t)}f^p(x)\mathrm{d}x, \tag{1.8}$$

$$\begin{aligned}&\int_0^b\left(\int_0^b\frac{f(x)}{(x+y)^t}\mathrm{d}x\right)^p\mathrm{d}y\\&\le B\left(\frac{1}{q},t-\frac{1}{q}\right)^{p-1}\int_0^b\left(B\left(\frac{1}{p}+\frac{p(1-t)}{q},p(t-1)+\frac{1}{q}\right)-\bar\theta_{p(1-t)+\frac{1}{p}}\left(\frac{x}{b}\right)^{p(t-1)+\frac{1}{q}}\right)\times\\&x^{p(1-t)}f^p(x)\mathrm{d}x,\end{aligned} \tag{1.9}$$

$$\begin{aligned}&\int_a^\infty\left(\int_a^\infty\frac{f(x)}{(x+y)^t}\mathrm{d}x\right)^p\mathrm{d}y\\&\le B\left(\frac{1}{q},t-\frac{1}{q}\right)^{p-1}\int_a^\infty\left(B\left(\frac{1}{p}+\frac{p(1-t)}{q},p(t-1)+\frac{1}{q}\right)-\bar\theta_{\frac{p(t-1)}{q}+\frac{1}{q}}\left(\frac{a}{x}\right)^{\frac{1}{p}+\frac{p(1-t)}{q}}\right)\times\\&x^{p(1-t)}f^p(x)\mathrm{d}x.\end{aligned} \tag{1.10}$$

Where

$$\begin{aligned}E=&\left[B\left(\frac{1}{q},t-\frac{1}{q}\right)-2\left(\bar\theta_{\frac{1}{p}}\bar\theta_{2-\frac{1}{p}-t}\left(\frac{a}{b}\right)^{t-\frac{1}{q}}\right)^{\frac{1}{2}}\right]^{p-1}\times\\&\left[B\left(\frac{1}{p}+\frac{p(1-t)}{q},p(t-1)+\frac{1}{q}\right)-2\left(\bar\theta_{\frac{p(t-1)}{q}+\frac{1}{q}}\bar\theta_{p(1-t)+\frac{1}{p}}\left(\frac{a}{b}\right)^{p(t-1)+\frac{1}{q}}\right)^{\frac{1}{2}}\right],\end{aligned}$$

$$F=\left[B\left(\frac{1}{q},t-\frac{1}{q}\right)-2\left(\overline{\theta}_{\frac{1}{p}}\overline{\theta}_{2-\frac{1}{p}-t}\left(\frac{a}{b}\right)^{t-\frac{1}{q}}\right)^{\frac{1}{2}}\right]^{p-1}\times$$

$$\left[B\left(\frac{1}{p}+\frac{p(1-t)}{q},p(t-1)+\frac{1}{q}\right)-2\left(\overline{\theta}_{\frac{p(t-1)}{q}+\frac{1}{q}}\overline{\theta}_{p(1-t)+\frac{1}{p}}\left(\frac{a}{b}\right)^{\frac{1}{p}+\frac{p(1-t)}{q}}\right)^{\frac{1}{2}}\right],$$

$$G=\left[B\left(\frac{1}{q},t-\frac{1}{q}\right)-2\left(\overline{\theta}_{\frac{1}{p}}\overline{\theta}_{2-\frac{1}{p}-t}\left(\frac{a}{b}\right)^{\frac{1}{q}}\right)^{\frac{1}{2}}\right]^{p-1}\times$$

$$\left[B\left(\frac{1}{p}+\frac{p(1-t)}{q},p(t-1)+\frac{1}{q}\right)-2\left(\overline{\theta}_{\frac{p(t-1)}{q}+\frac{1}{q}}\overline{\theta}_{p(1-t)+\frac{1}{p}}\left(\frac{a}{b}\right)^{p(t-1)+\frac{1}{q}}\right)^{\frac{1}{2}}\right],$$

$$H=\left[B\left(\frac{1}{q},t-\frac{1}{q}\right)-2\left(\overline{\theta}_{\frac{1}{p}}\overline{\theta}_{2-\frac{1}{p}-t}\left(\frac{a}{b}\right)^{\frac{1}{q}}\right)^{\frac{1}{2}}\right]^{p-1}\times$$

$$\left[B\left(\frac{1}{p}+\frac{p(1-t)}{q},p(t-1)+\frac{1}{q}\right)-2\left(\overline{\theta}_{\frac{p(t-1)}{q}+\frac{1}{q}}\overline{\theta}_{p(1-t)+\frac{1}{p}}\left(\frac{a}{b}\right)^{\frac{1}{p}+\frac{p(1-t)}{q}}\right)^{\frac{1}{2}}\right],$$

$$\overline{\theta}_s=\int_0^1\frac{1}{(1+u)^t}\left(\frac{1}{u}\right)^s\mathrm{d}u,\ s=\frac{1}{p},2-\frac{1}{p}-t,\frac{p(t-1)+1}{q},p(1-t)+\frac{1}{p}.$$

$B(m,n)$ $(m,n>0)$ and f are indicated as in Theorem 1.

Proofs of theorems

First we prove the following Lemmas.

Lemma 1 *Let $p>1,\frac{1}{p}+\frac{1}{q}=1$ and define k_s and θ_s as*

$$k_s=\int_0^\infty\frac{1}{1+u}\left(\frac{1}{u}\right)^s\mathrm{d}u\quad\theta_s=\int_0^1\frac{1}{1+u}\left(\frac{1}{u}\right)^s\mathrm{d}u,$$

then $k_s=B(s,1-s)=\theta_{1-s}+\theta_s=k_{1-s}$. Where $s=\frac{1}{p},\frac{1}{q}$, $B(m,n)$ $(m,n>0)$ is indicated as in Theorem 1.

Lemma 2 *Let $p>1,\frac{1}{p}+\frac{1}{q}=1$, and define $h_s(y)$ as*

$$h_s(y)=y^{s-1}\int_0^y\frac{1}{1+u}\left(\frac{1}{u}\right)^s\mathrm{d}u\quad\text{for}\quad y\in(0,1],$$

then

$$h_s(y)\geq h_s(1)=\theta_s\quad(y\in(0,1]).\tag{2.11}$$

The equality contained in (2.11) holds only when $y=1$. Where $s=\frac{1}{p},\frac{1}{q}$.

Proof Fixed s, we have

$$h_s'(y) = -y^{s-2}\int_0^y \frac{u^{1-s}}{(1+u)^2}\mathrm{d}u < 0.$$

Then $h_s(y)$ is strictly decreasing on $(0,1]$. Hence $h_s(y) \geq h_s(1) = \theta_s$,and the equality holds only when $y = 1$. This proves the Lemma 2.

Lemma 3 *Let $b > a > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and define the weight function $\omega_s(a,b,x)$ as*

$$\omega_s(a,b,x) = \int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^s \mathrm{d}y \quad \text{for} \quad x \in [a,b],$$

then

$$\omega_s(a,b,x) < \begin{cases} B(s,1-s) - 2\left(\theta_s\theta_{1-s}\left(\frac{a}{b}\right)^s\right)^{\frac{1}{2}} & \text{for } 2s-1 \geq 0, \\ B(s,1-s) - 2\left(\theta_s\theta_{1-s}\left(\frac{a}{b}\right)^{1-s}\right)^{\frac{1}{2}} & \text{for } 2s-1 \leq 0, \end{cases} \tag{2.12}$$

$$\omega_s(0,b,x) = \lim_{a\to 0}\omega_s(a,b,x) \leq B(s,1-s) - \theta_{1-s}\left(\frac{x}{b}\right)^s \quad (x \in (0,b]), \tag{2.13}$$

$$\omega_s(a,\infty,x) = \lim_{a\to\infty}\omega_s(a,b,x) \leq B(s,1-s) - \theta_s\left(\frac{a}{x}\right)^{1-s} \quad (x \in [a,\infty)). \tag{2.14}$$

Where the constant θ_s is indicated as in Theorem 1, and $s = \frac{1}{p}, \frac{1}{q}$.

Proof Putting $u = \frac{y}{x}$, we have

$$\begin{aligned} \omega_s(a,b,x) &= \int_{a/x}^{b/x} \frac{1}{1+u}\left(\frac{1}{u}\right)^s \mathrm{d}u \\ &= k_s - \left[\int_0^{a/x} \frac{1}{1+u}\left(\frac{1}{u}\right)^s \mathrm{d}u + \int_{b/x}^{\infty} \frac{1}{1+u}\left(\frac{1}{u}\right)^s \mathrm{d}u\right] \\ &= k_s - \left[\int_0^{a/x} \frac{1}{1+u}\left(\frac{1}{u}\right)^s \mathrm{d}u + \int_0^{x/b} \frac{1}{1+v}\left(\frac{1}{v}\right)^s \mathrm{d}v\right] \\ &= k_s - \left[\left(\frac{a}{x}\right)^{1-s} h_s\left(\frac{a}{x}\right) + \left(\frac{x}{b}\right)^s h_{1-s}\left(\frac{x}{b}\right)\right], \end{aligned} \tag{2.15}$$

and k_s is indicated as in Lemma 1. By Lemma 1,2, we have

$$\begin{aligned} \omega_s(a,b,x) &< k_s - \left[\left(\frac{a}{x}\right)^{1-s} h_s\left(\frac{a}{x}\right) + \left(\frac{x}{b}\right)^s h_{1-s}\left(\frac{x}{b}\right)\right] = k_s - \left[\left(\frac{a}{x}\right)^{1-s}\theta_s + \left(\frac{x}{b}\right)^s\theta_{1-s}\right] \\ &\leq B(s,1-s) - 2\left[\theta_s\theta_{1-s}\left(\frac{a}{x}\right)^{1-s}\left(\frac{x}{b}\right)^s\right]^{1/2} = B(s,1-s) - 2\left[\theta_s\theta_{1-s}\frac{a^{1-s}}{b^s}x^{2s-1}\right]^{1/2}. \end{aligned}$$

When $2s-1 \geq 0$, we have $\omega_s(a,b,x) < B(s,1-s) - 2\left[\theta_s\theta_{1-s}\left(\frac{a}{b}\right)^s\right]^{1/2}$.

When $2s-1\leq 0$, we have $\omega_s(a,b,x) < B(s,1-s) - 2\left[\theta_s\theta_{1-s}\left(\frac{a}{b}\right)^{1-s}\right]^{1/2}$. Relation (2.12) is valid. By (2.15) and Lemma 1,2, as $a\to 0$ and $a\to\infty$,(2.13) and (2.14) can be proved. This is the proof.

Using the same method, we can prove the fiollowing Lemmas 4-6.

Lemma 4 *Let $p>1, \frac{1}{p}+\frac{1}{q}=1, 1-\frac{1}{pq}<t\leq 1$, and define $\overline{k}_s=\int_0^\infty \frac{1}{(1+u)^t}\left(\frac{1}{u}\right)^s du$ and $\overline{\theta}_s$ as in Theorem 2, then $\overline{k}_s = B(1-s,s+t-1)=\overline{\theta}_s+\overline{\theta}_{2-s-t}=\overline{k}_{2-s-t}$. Where $s=\frac{1}{p}, 2-\frac{1}{p}-t, \frac{p(t-1)+1}{q}, p(1-t)+\frac{1}{p}, B(m,n)$ $(m,n>0)$ is indicated as in Theorem 1.*

Lemma 5 *Let $p>1, \frac{1}{p}+\frac{1}{q}=1, 1-\frac{1}{pq}<t\leq 1$, and define $\overline{h}_s(y)$ as*

$$\overline{h}_s(y) = y^{s-1}\int_0^y \frac{1}{(1+u)^t}\left(\frac{1}{u}\right)^s du \quad \text{for} \quad y\in(0,1],$$

then

$$\overline{h}_s(y)\geq \overline{h}_s(1)=\overline{\theta}_s \quad (y\in(0,1]). \tag{2.16}$$

The equality contained in (2.16) holds only when $y=1$. Where $s=\frac{1}{p}, 2-\frac{1}{p}-t, \frac{p(t-1)+1}{q}, p(1-t)+\frac{1}{p}$.

Lemma 6 *Let $b>a>0, p>1, \frac{1}{p}+\frac{1}{q}=1, 1-\frac{1}{pq}<t\leq 1$, and define the weight function $\overline{\omega}_s(a,b,x)$ as*

$$\overline{\omega}_s(a,b,x) = \int_a^b \frac{1}{(x+y)^t}\left(\frac{x}{y}\right)^s dy \quad \text{for} \quad x\in[a,b],$$

then

$$\overline{\omega}_s(a,b,x) < \begin{cases} x^{1-t}\left[B(1-s,t+s-1) - 2\left(\overline{\theta}_s\overline{\theta}_{2-s-t}\left(\frac{a}{b}\right)^{t+s-1}\right)^{\frac{1}{2}}\right] & \text{for } t+2s\geq 2, \\ x^{1-t}\left[B(1-s,t+s-1) - 2\left(\overline{\theta}_s\overline{\theta}_{2-s-t}\left(\frac{a}{b}\right)^{1-s}\right)^{\frac{1}{2}}\right] & \text{for } t+2s\leq 2, \end{cases} \tag{2.17}$$

$$\overline{\omega}_s(0,b,x) = \lim_{a\to 0}\overline{\omega}_s(a,b,x) \leq x^{1-t}\left[B(1-s,t+s-1) - \overline{\theta}_{2-s-t}\left(\frac{x}{b}\right)^{t+s-1}\right], \tag{2.18}$$

$$\overline{\omega}_s(a,\infty,x) = \lim_{a\to\infty}\overline{\omega}_s(a,b,x) \leq x^{1-t}\left[B(1-s,t+s-1) - \overline{\theta}_s\left(\frac{a}{x}\right)^{1-s}\right]. \tag{2.19}$$

Where $s=\frac{1}{p}, 2-\frac{1}{p}-t, \frac{p(t-1)+1}{q}, p(1-t)+\frac{1}{p}$.

Proof of Theorem 1. We first prove relation (1.5) for $q\geq p$.

By Hölder's inequality and Lemma 3, for $y\in[a,b]$, we have

$$\left(\int_a^b \frac{f(x)}{x+y}dx\right)^p = \left(\int_a^b \frac{1}{(x+y)^{1/p}}\left(\frac{x}{y}\right)^{1/pq} f(x)\frac{1}{(x+y)^{1/q}}\left(\frac{y}{x}\right)^{1/pq}dx\right)^p$$
$$\leq \int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/q} f^p(x)dx\left(\int_a^b \frac{1}{x+y}\left(\frac{y}{x}\right)^{1/p}dx\right)^{p/q}$$

$$= \omega_{1/p}(a,b,y)^{p/q} \int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/q} f^p(x)\mathrm{d}x$$
$$\le \left(B\left(\frac{1}{p},\frac{1}{q}\right) - 2\left(\theta_{\frac{1}{p}}\theta_{\frac{1}{q}}\left(\frac{a}{b}\right)^{1/p}\right)^{\frac{1}{2}}\right)^{p/q} \int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/q} f^p(x)\mathrm{d}x,$$

then

$$\int_a^b \left(\int_a^b \frac{f(x)}{x+y}\mathrm{d}x\right)^p \mathrm{d}y \le \left(B\left(\frac{1}{p},\frac{1}{q}\right) - 2\left(\theta_{\frac{1}{p}}\theta_{\frac{1}{q}}\left(\frac{a}{b}\right)^{1/p}\right)^{\frac{1}{2}}\right)^{p/q} \int_a^b\int_a^b \frac{1}{x+y}\left(\frac{x}{y}\right)^{1/q} f^p(x)\mathrm{d}x\mathrm{d}y$$
$$= \left(B\left(\frac{1}{p},\frac{1}{q}\right) - 2\left(\theta_{\frac{1}{p}}\theta_{\frac{1}{q}}\left(\frac{a}{b}\right)^{1/p}\right)^{\frac{1}{2}}\right)^{p/q} \int_a^b \omega_{1/q}(a,b,x) f^p(x)\mathrm{d}x$$
$$\le \left(B\left(\frac{1}{p},\frac{1}{q}\right) - 2\left(\theta_{\frac{1}{p}}\theta_{\frac{1}{q}}\left(\frac{a}{b}\right)^{1/p}\right)^{\frac{1}{2}}\right)^{p} \int_a^b f^p(x)\mathrm{d}x.$$

Similarly, relation (1.5) for $q \le p$ can be proved.Using the same method, we can show that (1.6) and (1.7) are valid. Theorem 1 is proved.

In a similar way to the proof of Theorem 1, Theorem 2 can be showed.

Remark 1 When $p = q = 2$, (1.5),(1.6),(1.7) change to (1.2),(1.3),(1.4) respectively, it follows that Theorem 1 is a generalization of (1.1),(1.2),(1.3),(1.4).

Remark 2 When $t = 1$, Theorem 2 changes to Theorem 1, it is obvious that Theorem 2 is a generalization of Theorem 1.

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关于一类 Hilbert 积分不等式的推广

吕 中 学[1], 谢 鸿 政[2]

(1. 徐州师范大学工学院基础部, 江苏 徐州 221011;

2. 哈尔滨工业大学数学系, 黑龙江 哈尔滨 150001)

摘 要: 本文通过引入参数 p,q,a,b 和 t, 推广了 Hilbert 积分不等式的等价形式.