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The Pointwise Estimate for Stancu-Sikkema Operators *

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Abstract: In this paper, we construct Stancu-Sikkema operators and obtain the pointwise direct and converse theorems in terms of pointwise modulus of smoothness. Some results about Bernstein-Sikkema operators and Stancu operators are extended.

Key words: Stancu-Sikkema operators; modulus of smoothness; K-functional.

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1. Introduction

Bernstein-Sikkema operators are defined by

$$C_n(f,x)=\sum_{k=0}^{n}p_{n,k}(x)f(\frac{k}{n+\alpha(n)}),\qquad 0\le \alpha(n)\le q,\ q>0,$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ ($C_n(f,x)$ denotes Bernstein operators when $\alpha(n)=0$). Li[1] studied the operators and obtained the following results.

Theorem A *Let $f\in C[0,1]$, $4\le n\in N$. Then*

$$\|C_nf-f\|_\infty\le C\left(\int_{1/\sqrt{n}}^{1/2}\frac{\omega_\varphi^2(f,t)}{t^3}\mathrm{d}t+E_n(f)\right)/n.$$

Theorem B *Let $1>\beta>0$. Then* $\omega_\varphi^2(f,1/\sqrt{n})\le Cn^{-1}\left(\sum_{k=1}^{n}(\frac{n}{k})^\beta\|C_kf-f\|+n^\beta\|f\|\right)$.

Stancu operators were defined in [2] by

$$L_n(f,x)=\sum_{k=0}^{n}b_{n,k,s}(x)f(\frac{k}{n})\quad x\in I=[0,1],$$

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where $s \in N, 0 \le s < n/2, b_{n,k,s}(x) = \begin{cases} (1-x)p_{n-s,k}(x), & 0 \le k < s, \\ (1-x)p_{n-s,k}(x) + xp_{n-s,k-s}(x), & s \le k \le n-s, \\ xp_{n-s,k-s}(x), & n-s < k \le n. \end{cases}$

When $s = 0$ or $s = 1$, Stancu operators are Bernstein operators.

The properties of the operators were given in [3],[4]. In [5] Ditzian gave a direct theorem for Bernstein operators by using a new modulus of smoothness $\omega^2_{\varphi^\lambda}(f,t)(0 \le \lambda \le 1)$. His results contained the results of the classical modulus of smoothness $\omega^2(f,t)$ and Ditzian-Totik modulus $\omega^2_\varphi(f,t)$. Using the new modulus of smoothness, we in [6],[7] studied the pointwise approximation theories for some positive linear operators.

In this paper, we will construct the Stancu-Sikkema operators and study the pointwise direct and converse theorems.

Definition *Let $f \in C[0,1]$, Stancu-Sikkema operators are defined by*

$$M_n(f,x) = \sum_{k=0}^{n} b_{n,k,s}(x) f\left(\frac{k}{n+\alpha(n)}\right), \quad f(x) \in C[0,1], \tag{1.1}$$

where the definitions of $b_{n,k,s}(x)$ and $\alpha(n)$ are same as above.

Obviously, when $\alpha(n) = 0$, $M_n(f,x)$ denotes Stancu operators; When $s = 0$ or $s = 1$, $M_n(f,x)$ denotes Bernstein-Sikkema operators. By simple computation, we obtain that $M_n(1,x) = 1$, $M_n(t-x,x) = -\frac{\alpha(n)x}{n+\alpha(n)}$, $M_n((t-x)^2,x) = \frac{(n-s+s^2)\varphi^2(x)+\alpha^2(n)x^2}{(n+\alpha(n))^2}$, $M_n(f,x) = \sum_{k=0}^{n-s}[(1-x)f(\frac{k}{n+\alpha(n)}) + xf(\frac{k+s}{n+\alpha(n)})]p_{n-s,k}(x)$.

The following definitions are needed in this paper.

$$\omega_1(f,t) = \sup_{0<h\le t}\{\|\Delta^1_h f(x)\|\}; \quad \omega^2_{\varphi^\lambda}(f,t) = \sup_{0<h\le t}\{\|\Delta^2_{h\varphi^\lambda} f(x)\|\};$$

$$\overline{K}^2_{\varphi^\lambda}(f,t^2) = \inf_{g\in D^2_\lambda}\{\|f-g\| + t^2\|\varphi^{2\lambda}g''\| + t^{\frac{4}{2-\lambda}}\|g''\|\}; \quad \|f\|_0 = \sup_{x\in(0,1)}|\delta_n^{\alpha(\lambda-1)}(x)f(x)|;$$

$$\|f\|_2 = \sup_{x\in(0,1)}|\delta_n^{2+\alpha(\lambda-1)}(x)f''(x)|; \quad K^2_{\alpha,\lambda}(f,t^2) = \inf_{g\in C^2_{\alpha,\lambda}}\{\|f-g\|_0 + t^2\|g\|_2\}.$$

where $D^2_\lambda = \{f \in C[0,1] | f' \in A.C., \varphi^{2\lambda}f'' \in L_\infty[0,1]\}$, $\delta_n(x) = \max\{\varphi(x), \frac{1}{\sqrt{n+\alpha(n)}}\}$, $C^0_{\alpha,\lambda} = \{f \in C[0,1], f(0) = f(1) = 0, \|f\|_0 < +\infty\}$, $C^2_{\alpha,\lambda} = \{f \in C^0_{\alpha,\lambda}, f' \in A.C._{\text{loc}}, \|f\|_2 < +\infty\}$.

From [8, p.11, p.24], we know that $\omega^2_{\varphi^\lambda}(f,t) \sim \overline{K}^2_{\varphi^\lambda}(f,t^2)$.

In this paper, we will get the following main results.

Theorem 1 *Let $f(x) \in C[0,1]$, $0 \le \lambda \le 1$, $u_{n,s} = \frac{n-s+s^2}{(n+\alpha(n))^2}$, then*

$$|M_n(f,x) - f(x)| \le C\omega^2_{\varphi^\lambda}(f, \delta_n^{1-\lambda}(x)\sqrt{u_{n,s}}) + C\omega_1(f, \frac{\alpha(n)x}{n+\alpha(n)}).$$

Theorem 2 *Let $f(x) \in C[0,1]$, $0 \le \lambda \le 1$, $0 < \alpha < 2$, then*

$$K^2_{\alpha,\lambda}(f,1/n) \le Cn^{-1}\sum_{k=1}^{n}\|M_k f - f\|_0.$$

Theorem 3 *Let $f \in C[0,1]$, $0 \le \lambda \le 1$, $0 < \alpha < \frac{2}{2-\lambda}$, $\alpha(n) \neq 0$, $u_{n,s} = \frac{n-s+s^2}{(n+\alpha(n))^2}$, then*

$$|M_n(f,x) - f(x)| = O((\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))^\alpha) \Longleftrightarrow \omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha).$$

Remark 1 Let $\lambda = 1$. It is obvious that Theorem 2 contains Theorem B. Using [8, p25] $\omega^r(f, t^{1/1-\beta}) \le C\omega^r_\varphi(f,t)$ $(\beta = 1/2)$ and [8, p43] $\omega^r(f,t) \le Ct^r \left(\int_t^C \frac{\omega^{r+1}(f,u)}{u^{r+1}}\mathrm{d}u + \|f\|\right)$, we get $\omega^2_\varphi(f, \sqrt{u_{n,s}}) + \omega_1(f, 1/n) \le C\left(\int_{1/\sqrt{n}}^{1/2} \frac{\omega^2_\varphi(f,t)}{t^3}\mathrm{d}t + E_n(f)\right)/n$. Thus, Theorem 1 also consists of Theorem A.

Remark 2 Let $\lambda = 0$. Similar to the discussion of Bernstein-Durrmeyer operators (see [9]), we get that there is no continuous function $\psi(x)$ satisfying $\omega^2(f,h) = O(h^\alpha) \Longleftrightarrow$ $|M_n(f,x) - f(x)| \le C\psi(x)$ for $1 < \alpha < 2$.

Throughout this paper, C denotes a constant independent of n and x, which is not necessarily the same at each occurrence.

2. Lemmas

Lemma 1 *If $f \in C^2_{\alpha,\lambda}$, $0 < \alpha < 2$, then*

$$\|M_n f\|_2 \le Cn\|f\|_0, \qquad \|M_n'' f\| \le Cn^{2-\alpha(1-\lambda/2)}\|f\|_0,$$
$$\|M_n'' f\| \le \|f''\|, \qquad \|M_n f\|_2 \le \|f\|_2 + Cn^{\alpha(1-\lambda)/2-1}\|f''\|.$$

Proof Since

$$\begin{aligned} M_n''(f,x) =& (n-s)\{(n-s-1)\sum_{k=0}^{n-s-2}[(1-x)\Delta^2_{\frac{1}{n+\alpha(n)}} f(\frac{k+1}{n+\alpha(n)}) + x\Delta^2_{\frac{1}{n+\alpha(n)}} f(\frac{k+s+1}{n+\alpha(n)})]\times \\ & p_{n-s-2,k}(x) + 2\sum_{k=0}^{n-s-1}\Delta_{\frac{1}{n+\alpha(n)}}(\Delta_{\frac{s}{n+\alpha(n)}}(f(\frac{k}{n+\alpha(n)}))p_{n-s-1,k}(x)\}, \end{aligned}$$

$$\begin{aligned} M_n''(f,x) =& \sum_{k=0}^{n-s}\{[(1-x)p''_{n-s,k}(x) - 2p'_{n-s,k}(x)]f(\frac{k}{n+\alpha(n)}) + \\ & (xp''_{n-s,k}(x) + 2p'_{n-s,k}(x))f(\frac{k+s}{n+\alpha(n)})\}, \end{aligned}$$

we can prove Lemma 1 by simple computation or by the method of Lemma 2.1 in [7].

Lemma 2 *Suppose that $n \in N$, $1 \le k \le n$, $l > 0$, $0 < r < l$ and the non-negative sequences $\{\nu_n\}$,$\{\psi_n\}$, $\{\mu_n\}$ satisfy $\nu_1 = \mu_1 = 0$. If $\nu_n \le (\frac{k}{n})^l\nu_k + C\psi_k$, $\mu_n \le (\frac{k}{n})^r\mu_k + C(\nu_k + \psi_k)$, then $\mu_n \le Cn^{-r}\sum_{k=1}^{n} k^{r-1}\psi_k$.*

The proof is similar to Lemma 2 in [10].

Lemma 3[6] *If $r \in N$, $0 < t < \frac{1}{8r}$, $rt/2 < x < 1 - rt/2$ and $0 < \beta \le r$, then*

$$\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\delta_n^{-\beta}(x + \sum_{i=1}^{r}u_i)\mathrm{d}u_1\cdots\mathrm{d}u_r \le Ct^r\delta_n^{-\beta}(x)$$

3. Proof of Theorems

Proof of Theorem 1 By the definition of $M_n(f,x)$, we have $\|M_nf\| \le \|f\|$.

Let $T_n(f,x) = f(x + \frac{\alpha(n)x}{n+\alpha(n)}) - f(x)$, $A_n(f,x) = M_n(f,x) + T_n(f,x)$, then

$$A_n(1,x) = 1, \quad A_n(t-x,x) = 0, \quad \|A_nf\| \le 3\|f\|. \tag{3.1}$$

From the definition of $\overline{K}^2_{\varphi^\lambda}(f,t^2)$, we can choose $g \in D^2_\lambda$ such that

$$\begin{aligned}&\|f-g\| + (\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))^2\|\varphi^{2\lambda}g''\| + (\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))^{\frac{4}{2-\lambda}}\|g''\|\\ &\le C\overline{K}^2_{\varphi^\lambda}(f,u_{n,s}\delta_n^{2(1-\lambda)}(x)).\end{aligned} \tag{3.2}$$

Applying Taylor formula, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\mathrm{d}u. \tag{3.3}$$

By $\frac{|t-u|}{\delta_n^{2\lambda}(u)} \le \frac{|t-x|}{\delta_n^{2\lambda}(x)}$ for u between t and x (see [8, p141]), (3.1) and (3.3), we have

$$\begin{aligned}|A_n(g,x)-g(x)| &= |A_n(\int_x^t (t-u)g''(u)\mathrm{d}u, x)|\\ &\le |M_n(\int_x^t (t-u)g''(u)\mathrm{d}u, x)| + |\int_x^{x+\frac{\alpha(n)x}{n+\alpha(n)}} (x + \frac{\alpha(n)x}{n+\alpha(n)} - u)g''(u)\mathrm{d}u|\\ &\le \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda}g''\|[M_n((t-x)^2,x) + \frac{\alpha^2(n)x^2}{(n+\alpha(n))^2}] \le C\delta_n^{2-2\lambda}(x)u_{n,s}\|\delta_n^{2\lambda}g''\|.\end{aligned} \tag{3.4}$$

Similarly, by $\frac{|t-x|}{\varphi^{2\lambda}(u)} \le \frac{|t-x|}{\varphi^{2\lambda}(x)}$ (see [8, p141]), we have

$$|A_n(g,x)-g(x)| \le C\delta_n^2(x)u_{n,s}\varphi^{-2\lambda}(x)\|\varphi^{2\lambda}g''\|. \tag{3.5}$$

If $x \in E_n = [1/(n+\alpha(n)), 1-1/(n+\alpha(n))]$, then $\delta_n(x) = \varphi(x)$. From (3.2) and (3.5), we get

$$\begin{aligned}|A_n(f,x)-f(x)| &\le C[\|f-g\| + u_{n,s}\delta_n^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\|]\\ &\le C\overline{K}^2_{\varphi^\lambda}(f,u_{n,s}\delta_n^{2(1-\lambda)}(x)) \le C\omega^2_{\varphi^\lambda}(f,\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x)).\end{aligned}$$

If $x \in E_n^c = [0,1/(n+\alpha(n)))\bigcup(1-1/(n+\alpha(n)),1]$, then $\delta_n(x) = \frac{1}{\sqrt{n+\alpha(n)}}$, from (3.2) and (3.4), we obtain

$$\begin{aligned}&|A_n(f,x)-f(x)|\\ &\le C\{\|f-g\| + u_{n,s}\delta_n^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\| + u_{n,s}\delta_n^{2(1-\lambda)}(x)(n+\alpha(n))^{-\lambda}\|g''\|\}\\ &\le C\{\|f-g\| + u_{n,s}\delta_n^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\| + (\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))^{\frac{4}{2-\lambda}}\|g''\|\}\\ &\le C\overline{K}^2_{\varphi^\lambda}(f,u_{n,s}\delta_n^{2(1-\lambda)}(x)) \le C\omega^2_{\varphi^\lambda}(f,\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x)).\end{aligned}$$

Thus

$$|M_n(f,x)-f(x)| \le |A_n(f,x)-f(x)|+|T_n(f,x)| \le C\omega^2_{\varphi^\lambda}(f,\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))+\omega_1(f,\frac{\alpha(n)x}{n+\alpha(n)}). \tag{3.6}$$

Proof of Theorem 2 Let $\mu_n = n^{-1}\|M_n f\|_2$, $\nu_n = n^{-2+\alpha(1-\lambda)/2}\|M''_n f\|$, $\psi_n = \|M_n f - f\|_0$, then $\mu_1 = \nu_1 = 0$. Similar to the proof of [10, Theorem 2.2], the theorem follows by Lemma 1 and Lemma 2.

Proof of Theorem 3 "$\Longrightarrow$" Since $M_n(f,x)$ preserve constants, it is sufficient to prove the theorem for $f \in C^0_{\alpha,\lambda}$.

If $|M_n(f,x) - f(x)| = O((\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))^\alpha)$, then $\|M_n(f,x) - f(x)\|_0 = O((\sqrt{u_{n,s}})^\alpha)$. By Theorem 2, we have $K^2_{\alpha,\lambda}(f,n^{-1}) \le Cn^{-1}\sum\limits_{k=1}^{n}(u_{k,s})^{\alpha/2} \le Cn^{-\alpha/2}$.

Choosing $\frac{1}{n+1} < t \le \frac{1}{n}$, we get

$$K^2_{\alpha,\lambda}(f,t^{-1}) \le Ct^{-\alpha/2}. \tag{3.7}$$

For $f \in C^0_{\alpha,\lambda}$, we have

$$|\Delta^2_{t\varphi^\lambda}f(x)| \le \|f\|_0[\delta_n^{\alpha(1-\lambda)}(x+t\varphi^\lambda(x))+2\delta_n^{\alpha(1-\lambda)}(x)+\delta_n^{\alpha(1-\lambda)}(x-t\varphi^\lambda(x))] \le 4\delta_n^{\alpha(1-\lambda)}(x)\|f\|_0. \tag{3.8}$$

From Lemma 3, for $g \in C^2_{\alpha,\lambda}$, $1-t \ge x \ge t$, we have

$$|\Delta^2_{t\varphi^\lambda}g(x)| \le |\int_{-t\varphi^\lambda/2}^{t\varphi^\lambda/2}\int_{-t\varphi^\lambda/2}^{t\varphi^\lambda/2} g''(x+u_1+u_2)\mathrm{d}u_1\mathrm{d}u_2| \le C\|g\|_2 t^2\delta_n^{(2-\alpha)(\lambda-1)}(x). \tag{3.9}$$

According to the definition of $K^2_{\alpha,\lambda}(f,t)$, we can choose a proper $g \in C^2_{\alpha,\lambda}$ such that

$$\|f-g\|_0 + t^2\|g\|_2 \le 2K^2_{\alpha,\lambda}(f,t^2). \tag{3.10}$$

Thus by (3.7)-(3.10), we have

$$\begin{aligned}|\Delta^2_{t\varphi^\lambda}f(x)| &\le |\Delta^2_{t\varphi^\lambda}(f-g)(x)| + |\Delta^2_{t\varphi^\lambda}g(x)|\\ &\le C\delta_n^{\alpha(1-\lambda)}(x)\|f-g\|_0 + Ct^2\delta_n^{(2-\alpha)(\lambda-1)}(x)\|g\|_2\\ &\le C\delta_n^{\alpha(1-\lambda)}(x)K^2_{\alpha,\lambda}(f,t^2\delta_n^{2(\lambda-1)}(x)) \le Ct^\alpha.\end{aligned}$$

That means $\omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha)$.

"$\Longleftarrow$" Using[8,p25] $\omega^r(f,t^{1/(1-\lambda/2)}) \le C\omega^r_{\varphi^\lambda}(f,t)$ and [8, p43]

$$\omega^r(f,t) \le Ct^r\left(\int_t^C \frac{\omega^{r+1}(f,u)}{u^{r+1}}\mathrm{d}u + \|f\|\right),$$

we get that $\omega_1(f,t) \le Ct^{(1-\lambda/2)\alpha}$ under the condition of $\omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha)$.

Thus, from Theorem 1, we have

$$\begin{aligned}|M_n(f,x)-f(x)| &\le C\{\omega^2_{\varphi^\lambda}(f,\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))+\omega_1(f,\frac{1}{n+\alpha(n)})\}\\ &\le C\{u_{n,s}^{\alpha/2}\delta_n^{(1-\lambda)\alpha/2}(x)+(n+\alpha(n))^{-(1-\lambda/2)\alpha}\}\le C(\sqrt{u_{n,s}}\delta_n^{1-\lambda}(x))^\alpha.\end{aligned}$$

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Stancu-Sikkema 算子的点态估计

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摘 要: 本文构造了 Stancu-Sikkema 算子, 并利用点态光滑模研究了此算子的点态逼近正逆定理.