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C^∞ Compactness for Minimal Submanifolds in the Unit Sphere *

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Abstract: In this paper we study the C^∞ compactness for minimal submanifolds in the unit sphere. We obtain two compactness theorems. As an application, we prove that there is a positive number δ (n), such that if the square of the length of the second fundamental form of a minimal submanifold in the unit sphere is less than $\frac{2}{3}n+\delta(n)$, it must be totally geodesic or diffeomorphic to a Veronese surface.

Key words: minimal submanifold; totally geodesic; compactness theorem.

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1. Introduction

The compactness theorem give us good understanding of the behavior of Riemannian manifolds under certain restricts of curvature, volume and diameter, see for example [2,6,7,11,8]. In [12], Shen obtained a $C^{1,\alpha}$convergence theorem for Riemannian submanifolds. However, $C^{1,\alpha}$ convergence is not enough for many applications. In this paper, we will study C^∞convergence for minimal submanifolds which are isometrically immersed or imbedded in the unit sphere. We should mention that Choi and Schoen[4] have obtained the C^∞compactness theorem for minimal surfaces in three dimensional manifolds.

Suppose $M^n$is a minimal n-dimensional submanifold of the $(n+p)$-dimensional unit sphere S^{n+p}. First let's fix some notations. We denoted by $V(M)$, $d(M)$, $s(M)$, $\mathrm{Ric}(M)$ and $K(M)$the volume, diameter, square of the length of the second fundamental form, Ricci curvature, and sectional curvature of M^nrespectively. Let $\Gamma_{n+p}=\{M^n|M^n$is a minimal closed Riemannian submanifold immersed in $S^{n+p}\}$. Here, we don't distinguish isometric manifolds. Given a sequence of Riemannian manifolds $\{M_i^n\}_{i=1}^\infty$, we say $\{M_i^n\}C^\infty$ converges to a Riemannian manifold M^n, if for all but finitely many i, there exist diffeomorphisms $f_i: M^n\to M_i^n$such that $\{f_i^*g_i\}C^k$ converges to g for every k, where g_i and $g$are the Riemannian metrics of M_i^n and M respectively. Now we can express our

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theorems as follows:

Theorem 1 *Let $\Gamma^s_{n+1} = \{M^n \in \Gamma_{n+1} | M^n$ be imedded in $S^{n+1}, s(M) \le s\}$. Then Γ^s_{n+1} contains only finitely many diffeomorphism types of n-manifolds. Moreover, Γ^s_{n+1} is C^∞ compact.*

One may hope to prove that $\{M^n \in \Gamma_{n+1} | s(M) \le s\}$ is C^∞ compact. However, this is not ture, because $s(M)$ can not control $V(M)$ if M^n is only immersed. For higher codimensional case, we have

Theorem 2 *Let $\Gamma^{V,s}_{n+p} = \{M^n \in \Gamma_{n+p} | V(M) \le V, s(M) \le s\}$. Then $\Gamma^{V,s}_{n+p}$ contains only finitely many diffeomorphism types of n-manifolds. Moreover, $\Gamma^{V,s}_{n+p}$ is C^∞ compact.*

As an interesting application of the above theorem, we have the following pinching theorem, which improves the previous pinching constant is [3,5,14].

Theorem 3 *For every $n \ge 2$, there is a positive number $\delta(n)$, such that if $s(M) \le \frac{2}{3}n + \delta(n)$, $M^n \in \Gamma_{n+p}$, then M^n must be totally geodesic or diffeomorphic to the Veronese surface.*

2. Some necessary estimates

To prove the above theorems, we need to estimate the bounds of curvature, volume and diameter of a minimal submanifold. Some facts are well known.

Lemma 1 *If $M^n \in \Gamma_{n+p}$, then $V(M) \ge V(S^n)$. Where $V(S^n)$ is the volume of the standard n-dimensional sphere.*

Proof This follows from [9].

Lemma 2 *If $M^n \in \Gamma^s_{n+1}$, then $V(M) \le C(s)$, where $C(s)$ is a constant depend only on s.*

Proof This follows from [10].

To estimate curvature and their derivatives, we shall introduce the structure equations of submanifold. Suppose $M^n \in \Gamma_{n+p}$. Choose local orthonormal frame $\{e_1, e_2, \cdots, e_{n+p}\}$for S^{n+p}, such that when restricted to M^n, $\{e_1, e_2, \cdots, e_n\}$is tangent to M^n.The dual $1-$forms are denoted by $\{\omega_1, \omega_2, \cdots, \omega_{n+p}\}$. We have the structure equations for S^{n+p}as follows:

$$d\omega_A = -\sum_{B=1}^{n+p} \omega_{AB} \wedge \omega_B, A \in \{1, 2, \cdots, n+p\}, \omega_{AB} = -\omega_{BA},$$
$$d\omega_{AB} = -\sum_{i=1}^{n+p} \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \ \ A, B \in \{1, 2, \cdots, n+p\},$$

where $\Omega_{AB} = \frac{1}{2}\sum_{C,D=1}^{n+p} R_{ABCD}\omega_C \wedge \omega_D, R_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}$.

When restricted to M^n, the above equations turn out to be

$$
\begin{aligned}
d\omega_i &= -\sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, i \in \{1,2,\cdots,n\}, \omega_{ij} = -\omega_{ji}, \\
d\omega_{ij} &= -\sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, i,j \in \{1,2,\cdots,n\}, \\
\Omega_{ij} &= \frac{1}{2}\sum_{k,l=1}^{n} K_{ijkl}\omega_k \wedge \omega_l, K_{ijkl} = R_{ijkl} = \sum_{\alpha}(h^{\alpha}_{ik}h^{\alpha}_{jl} - h^{\alpha}_{il}h^{\alpha}_{jk}),
\end{aligned} \tag{1}
$$

where h^{α}_{ij} satisfies

$$
\begin{aligned}
\omega_{\alpha i} &= \sum_{j=1}^{n} h^{\alpha}_{ij}\omega_j, h^{\alpha}_{ij} = h^{\alpha}_{ji}, \\
d\omega_{\alpha\beta} &= -\sum_{\gamma=n+1}^{n+p} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \\
\Omega_{\alpha\beta} &= \frac{1}{2}\sum_{kl} K_{\alpha\beta kl}\omega_k \wedge \omega_l, K_{\alpha\beta kl} = R_{\alpha\beta kl} + \sum_{i}(h^{\alpha}_{ik}h^{\beta}_{il} - h^{\alpha}_{il}h^{\beta}_{jk}).
\end{aligned}
$$

The second fundamental form of M^n is

$$H = \sum_{\alpha ij} h^{\alpha}_{ij}\omega_i \otimes \omega_j \otimes e_{\alpha}.$$

Then $s(M)$, the square of the length of H, is equal to $\sum_{\alpha ij}(h^{\alpha}_{ij})^2$.

Lemma 3 *For $M^n \in \Gamma_{n+p}$, we have*

$$|K(M)| \le 1 + 2s(M), \quad \mathrm{Ric}(M) \ge (n-1)[1 - s(M)/n].$$

Proof The first estimate follows from (1). Next we will verify the second estimate at every fixed point $x \in M^n$. Denoted by $H^{\alpha}$the $n \times n$ matrix whose (i,j) entry is h^{α}_{ij}. Set $G = \sum_{\alpha}(H^{\alpha})^2$. By choosing suitable frame $\{e_1, e_2, \cdots, e_n\}$ we can assume that G is diagonal at x. Thus by the minimality of M^n and (1), we have

$$\sum_{j} K_{ijkj} = (n-1)\delta_{ik} - G_{ik}. \tag{2}$$

Also, we have

$$
\begin{aligned}
n(h^{\alpha}_{ij})^2 &= (n-1)(h^{\alpha}_{ij})^2 + (-\sum_{j\neq i} h^{\alpha}_{jj})^2 \\
&\le (n-1)(h^{\alpha}_{ii})^2 + (n-1)\sum_{j\neq i}(h^{\alpha}_{jj})^2 = (n-1)\sum_{j}(h^{\alpha}_{jj})^2.
\end{aligned} \tag{3}
$$

Then by (2) and (3) , we know that $\sum_j K_{ijkj}=0$ for $i\neq k$, and

$$\sum_j K_{ijij}=(n-1)-\sum_{\alpha j}(h_{ij}^{\alpha})^2=(n-1)-\sum_{\alpha}[(h_{ii}^{\alpha})^2+\sum_{j\neq i}(h_{ij}^{\alpha})^2]$$
$$\geq (n-1)-\sum_{\alpha}\frac{n-1}{n}\sum_{ij}(h_{ij}^{\alpha})^2=(n-1)[1-s(M)/n].$$

Lemma 4 *If $M^n\in\Gamma_{n+p}^{V,s}$, then $d(n,p,s)\leq d(M)\leq D(n,p,s,V)$.*

Proof The lower bound follows from Lemma 1,3 and the Bishop-Gromov volume comparison theorem. The upper bound follows from Lemma 3 and a simple packing argument.

Lemma 5 *If $M^n\in\Gamma_{n+p}^{V,s}$, then*

$$\left|D^k Rm\right|\leq C_k(n,p,s,V), k\geq 1.$$

Where Rm is the curvature tensor of M^n, D^kRm is the k-th covariant derivative of Rm, $|\cdot|$ is the pointwise norm, $C_k(n,p,s,V)=C_k$ is a constant depend only on n,p,s and V.

To prove this lemma, we only need to estimate $|D_mH|$, the pointwise norm of m-th covariant derivative of the second fundamental form. We put

$$D^mH=\sum_{\alpha i_1 i_2\cdots i_{m+2}} h^{\alpha}_{i_1i_2\cdots i_{m+2}}\omega i_1\otimes\omega_{i_2}\cdots\otimes\omega i_{i_{m+2}}\otimes e_{\alpha},$$
$$|D^mH|^2=\sum_{\alpha i_1 i_2\cdots i_{m+2}}(h^{\alpha}_{i_1i_2\cdots i_{m+2}})^2.$$

We have the following Ricci identity

$$h^{\alpha}_{i_1\cdots i_s kl}-h^{\alpha}_{i_1\cdots i_s lk}=\sum_r\sum_{j=1}^{s}h^{\alpha}_{i_1\cdots i_{j-1}ri_{j+1}\cdots i_s}R_{ri_jkl}+\sum_{\beta}h^{\beta}_{i_1\cdots i_s}R_{\beta\alpha kl}. \tag{4}$$

The computation of [3] gives

$$\sum_k h^{\alpha}_{ijkk}=\sum_{nk}h^{\alpha}_{nk}R_{nijk}+\sum_{nk}R_{nkjk}+\sum_{\beta k}h^{\beta}_{ik}R_{\beta\alpha jk}. \tag{5}$$

More generally, we have

$$\sum_k h^{\alpha}_{i_1i_2\cdots i_m kk}=\sum_k(h^{\alpha}_{i_1i_2\cdots i_{m-1}i_mk}-h^{\alpha}_{i_1i_2\cdots i_{m-1}ki_m})_k+$$
$$\sum_k(h^{\alpha}_{i_1i_2\cdots ki_mk}-h^{\alpha}_{i_1i_2\cdots kki_m})+(\sum_k h^{\alpha}_{i_1i_2\cdots i_{m-1}kk})_{i_m}, m\geq 3. \tag{6}$$

Now we have

$$\frac{1}{2}\nabla|D^mH|^2=-\left|D^{m+1}H\right|^2-\sum_{\alpha i_1i_2\cdots i_{m+2}}h^{\alpha}_{i_1i_2\cdots i_{m+2}}\sum_k h^{\alpha}_{i_1i_2\cdots i_{m+2}kk}. \tag{7}$$

Where $\nabla$is the Laplacian $-\mathrm{tr}D^2$.

Proof of Lemma 5 By definition, $\left|D^0H\right|^2 = s(M) \le s$. Integrate (7) on M^n, using Stokes' formula and (6), we get $\int_{M^n} |DH|^2 \le d_1$.

Now suppose

$$\left|D^k H\right| \le C_k, k = 0, 1, 2, \cdots, m-1, m \ge 1.$$

$$\int_{M^n} |D^m H|^2 \le d_m.$$

From (4), (5), (6), and (7), we have

$$\frac{1}{2}\nabla |D^m H|^2 \le -\left|D^{m+1}H\right|^2 + a|D^m H|^2 + b|D^m H|.$$

Integrating gives

$$\int_{M^n} \left|D^{m+1}H\right|^2 \le d_{m+1}.$$

Now, from (8), we can deduce

$$\begin{aligned}|D^m H|\nabla|D^m H| \le &|d|D^m H||^2 - \left|D^{m+1}H\right|^2 + a|D^m H|^2 + b|D^m H| \\ \le &a|D^m H|^2 + b|D^m H|. \end{aligned} \tag{8}$$

We have applied Kato's inequality $|d|D^m H||^2 \le \left|D^{m+1}H\right|^2$. Rewrite (9) as

$$\nabla(|D^m H| + b/a) \le a(|D^m H| + b/a).$$

Then by Theorem 3 of Appendix five of [1], we get the estimate $|D^m H| + b/a \le C_m$. We should mention that Lemma 3, Lemma 4, Theorem 2 of Appendix one of [1], and Theorem 3 of Appendix four of [1] provide the condition needed by the theorem we used.

By induction we get the estimate $\left|D^k H\right| \le C_k$for all $k \ge 0$. Now by the structure equations, we also get the estimate of $\left|D^k Rm\right|$, which finishes the proof.

3. Proofs of the theorems

We are now in a position to prove Theorem 1, 2, and 3.

Proof of Theorem 1 and 2 First let's consider $\Gamma^{V,s}_{n+p}$. By lemma 1, 4 and 5 and the C^k version of the Cheeger-Gromov compactness theorem (cf. [6,8,11]), we know that $\Gamma^{V,s}_{n+p}$consists of only finitely many diffeomorphism types of n-manifolds. Moreover, given a sequence of $\{M^n_i\}$ which belong to $\Gamma^{V,s}_{n+p}$, we can choose a subsequence which C^∞ converges to a n-manifold M^n. Without loss of generality, we can assume that $\{M^n_i\}$ itself C^∞converges to M^n. Thus for i sufficiently large, there are diffeomorphisms $f_i : M^n \to M^n_i$, such that $f^*_i g_i C^\infty$ converges to a Riemannian metric g on M^n. Where $g_i = \varphi^*_i(ds^2)$ is induced by the isometric immersion $\varphi_i : M^n_i \to S^{n+p} \subset R^{n+p+1}$, ds^2 is the standard metric on R^{n+p+1}. We shall prove that $M^n \in \Gamma^{V,s}_{n+p}$ which will finish the proof of Theorem 2.

By Takahashi's theorem [13], $\nabla_{gi}\varphi_i = n\varphi_i$. Then by similar calculations of Lemma 5, we can find constants C_k which independ of i, such that $\left|D^k_{g_i}\varphi_i\right| \le C_k$. Thus

$$\left|D^k_{f_i^* g_i}(\varphi_i f_i)\right| = \left|D^k_{g_i}\varphi_i\right| \le C_k.$$

Now since $\{f_i^* g_i\}C^\infty$ converges to g, for i sufficiently large, we have

$$\left|D^k_g(\varphi_i f_i)\right| \le 2C_k.$$

Then by Arzela-Ascoli's theorem, there is subsequence of $\{\varphi_i f_i\}$ which C^∞ converges to a C^∞map $\varphi : M^n \to R^{n+p+1}$. It's apparent that $\varphi^*(ds^2) = g, \nabla_g\varphi = n\varphi$. Thus again by Takahashi's theorem [13], φ is a minimal isometric immersion which maps M^ninto S^{n+p}. Now it's easy to see that $M^n \in \Gamma^{V,s}_{n+p}$.

Next let's consider Γ^s_{n+1}. By Lemma 2 and the above proof, we know that Γ^s_{n+1} consists of finitely many diffeomorphism type of n-manifolds. Moreover, given a sequence of $\{M_i^n\} \in \Gamma^s_{n+1}$, we can find a subseqence C^∞ converges to M^n which is minimally immersed in S^{n+1}. We want to prove that $M^n \in \Gamma^s_{n+1}$. We will assume that it's $\{M_i^n\}$ itself that C^∞convergest to M^n. By the above proof, we assume that there are diffeomorphisms $f_i : M^n \to M_i^n$and isometric imbeddings $\varphi_i : M_i^n \to S^{n+1} \subset R^{n+2}$. It's only need to prove that φ is an imbedding.

Suppose on the contrary, φ is not imbedded. Then there are different points $x, y \in M^n$ such that $\varphi(x) = \varphi(y) = p \in S^{n+1}$. Since $\varphi$is an immersion, there are geodesic balls $B_x(\varepsilon), B_y(\varepsilon)$ such that $\varphi|B_x(2\varepsilon)$ and $\varphi|\, B_y(2\varepsilon)$ are imbeddings, $B_x(2\varepsilon) \cap B_y(2\varepsilon) = \varphi$. Note that these balls are selected with respect to the metric $\varphi^* g_0$, where g_0 is the standard metric on S^{n+1}. Now there is a positive real number δ such that

$$\mathrm{dist}_{g_0}(\varphi(\partial B_x(\varepsilon)), p) > \delta, \mathrm{dist}_{g_0}(\varphi(\partial B_y(\varepsilon)), p) > \delta.$$

Where dist_{g_0} is the distance function on S^{n+1}. Since $\{\varphi_i f_i\}C^\infty$ converges to φ, for sufficiently large i, we have

$$\mathrm{dist}_{g_0}(\varphi_i f_i(\partial B_x(\varepsilon)), p) > \frac{\delta}{2},\ \ \mathrm{dist}_{g_0}(\varphi_i f_i(\partial B_y(\varepsilon)), p) > \frac{\delta}{2}.$$

Put $x_i = \varphi_i f_i(x), y_i = \varphi_i f_i(y)$. Then $\lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = p$. So for sufficienctly large i, we have

$$\mathrm{dist}_{g_0}(x_i, p) < \min(\frac{\delta}{10}, \frac{\mathrm{arccot}\sqrt{s}}{4}),\ \mathrm{dist}_{g_0}(y_i, p) < \min(\frac{\delta}{10}, \frac{\mathrm{arccot}\sqrt{s}}{4}).$$

Now for every i, choose $z_i \in \varphi_i f_i(\overline{B_y(\varepsilon)})$, such that

$$\mathrm{dist}_{g_0}(x_i, \varphi_i f_i(\overline{B_y(\varepsilon)}) = \mathrm{dist}_{g_0}(x_i, z_i).$$

Then the following inequalities hold for i sufficiently large

$$\begin{aligned}\mathrm{dist}_{g_0}(p, z_i) &\le \mathrm{dist}_{g_0}(p, y_i) + \mathrm{dist}_{g_0}(y_i, x_i) + \mathrm{dist}_{g_0}(x_i, z_i)\\ &\le \mathrm{dist}_{g_0}(p, y_i) + 2\mathrm{dist}_{g_0}(y_i, x_i) \le 3\mathrm{dist}_{g_0}(p, y_i) + 2\mathrm{dist}_{g_0}(p, x_i) < \frac{\delta}{2}.\end{aligned}$$

This implies that $z_i \in \varphi_i f_i(B_y(\varepsilon))$. Choose a minimal geodesic γ_i in S^{n+1} which connects z_i and x_i. Then by the formula for the first variation of arclength, we know that $\dot{\gamma}_i(0) \in N(\varphi_i f_i(M^n))$, where $N(\varphi_i f_i(M^n))$ is the normal bundle of $\varphi_i f_i(M^n)$. Set $l_i = \mathrm{dist}_{g_0}(z_i, x_i)$. Then

$$\exp^{\perp}_{z_i}(l_i \dot{\gamma}_i(0)) = \exp^{\perp}_{x_i}(0),$$

where $\exp^{\perp}$ is the normal exponential map of $\varphi_i f_i(M^n)$. Now $l_i \leq \mathrm{dist}_{g_0}(y_i, x_i) < \frac{1}{2}\mathrm{arccot}\sqrt{s}$, which implies that $\exp^{\perp}$ is not injective on $\{v \in N(\varphi_i f_i\,(M^n))| \ \ \|v\|_{g_0} < \mathrm{arccot}\sqrt{s}\}$. But this contradicts with Lemma 3 of [10], so our proof is completed. □

Finally, we prove Theorem 3, which gives an application of the above results.

Proof of Theorem 3 Recall the theorems in [5] that, given $M^n \in \Gamma_{n+p}$, if $s(M) \leq \frac{2}{3}n$, then M^n must be totally geodesic or is the Veronese surface (in which case $n = 2, s = 4/3$). Suppose on the contrary the conclusion of Theorem 3 is not ture. Then there is $M_i^n \in \Gamma_{n+p}$such that $s(M_i) \leq \frac{2}{3}n + \varepsilon_i, \lim_{i\to\infty} \varepsilon_i = 0$, $M_i^n$is neither totally geodesic nor diffeomorphic to the Veronese surface. We can assume that $0 < \varepsilon_i < \frac{n}{12}$. By lemma 3 we know that $\mathrm{Ric}(M) \geq \frac{1}{4}(n-1)$. By the Bishop-Gromov volume comparison theorem, we have $V(M_i) \leq V(S^n(2))$, where $S^n(2)$ is the sphere in R^{n+1} with radius 2. So $M_i^n \in \Gamma^{V,s}_{n+p}$. Where $V = V(S^n(2)), S = \frac{3}{4}n$. Applying Theorem 2 we get a $M^n \in \Gamma^{V,s}_{n+p}$, such that a subsequence of $\{M_i^n\} C^\infty$ converges to M^n. Now it's easy to see that $s(M) \leq \frac{2}{3}n$. By the definition of C^∞convergence and the above assumption, M^n is not the Veronese surface. So by the pinching theorem mentioned at the beginning, M^nmust be totally geodesic. So $s(M) = 0$. But this implies that for $i$sufficiently large, $s(M_i) < \frac{2}{3}n$, which yields that $M_i$is totally geodesic. This is a contradiction. □

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单位球面中极小子流形的 C^∞ 紧性

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摘 要: 本文研究了单位球面中极小子流形的 C^∞ 紧性，并得到两个紧性定理. 作为应用，我们证明了存在正数 $\delta(n)$，如果单位球面中极小子流形的第 2 基本形式的长度平方小于 $\frac{2}{3}n+\delta(n)$, 则它必须是全测地的或微分同胚于 Veronese 曲面.

关键词: 极小子流形；全测地线；紧性定理.