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On the φ_0-Stability of Impulsive Comparison Differential System *

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Abstract: This paper mainly uses the piecewise continuous cone-valued Lyapunov's function to obtain the φ_0-stability of impulsive comparison differential system (2), and uses the comparison equation to obtain the stability of impulsive differential system (1).

Key words: impulsive differential system; φ_0-equistable; uniformly φ_0-stable; equi-asymptotically φ_0-stable; uniformly asymptotically φ-stable.

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1. Introducation

Let R^n denote the n-dimensional Euclidean space with any convenient norm $\|\cdot\|$, and (,) the scalar product. $R_+ = [0,\infty), R = (-\infty,\infty), R_+^n = \{u \in R^n : u_i \geq 0, i = 1,2,\cdots,n\}$.$PC[R_+ \times R^n, R^n]$ denotes the space of piecewise continuous functions mapping $R_+ \times R^n$ into R^n.

Definition 1 *A proper subset G of R^n is called a cone, if*
(i) $\lambda G \subset G, \lambda \geq 0$;
(ii) $G + G \subset G$;
(iii) $\overline{G} = G$;
(iv) $G^0 = \varphi$;
(v) $G \cap \{-G\} = \{0\}$,
where $\overline{G}$ and G^0 denote the closure and interior of G respectively, and ∂G denotes the boundary of G.

The order relation on R^n induced by the cone G is defined as follows:
Let $x, y \in G$, then $x \leq_G y$ iff $y - x \in G$ and $x <_{G^0} y$ iff $y - x \in G^0$.

Definition 2 *The set G^* is called the adjoint cone if $G^* = \{\varphi \in R^n : (\varphi, x) \geq 0, x \in G\}$ satisfies properties (i)-(v) of Definition 1.*

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$x \in \partial G$ iff $(\varphi, x) = 0$ for some $\varphi \in G_0^*, G_0 = G - \{0\}$.

Definition 3 *A function $g : D \to R^n (D \subset R^n)$ is said to be quasimonotone relative to the cone G, if $x, y \in D$ and $y - x \in \partial G$ implies that there exists $\varphi_0 \in G_0^*$ such that $(\varphi_0, y - x) = 0$, and $(\varphi_0, g(y) - g(x)) \geq 0$.*

Consider the impulsive differential system

$$\begin{cases} \frac{dx}{dt} = f(t, x), & t \neq t_k, \\ \Delta x = I_k(x(t_k)), & t = t_k, k = 1, 2, \cdots, \\ x(t_0^+) = x_0. \end{cases} \quad (1)$$

Where $f \in PC[R_+ \times R^m, R^m], I_k \in C[R^m, R^m]$. Define S_ρ by $S_\rho = \{x \in R^m : \|x\| < \rho, \rho > 0\}$.Let $G \subset R^n$ be a cone in $R^n, n \leq m$, and $V \in PC[R_+ \times S_\rho, G]$. For $(t, x) \in R_+ \times S_\rho, h > 0$, define the function $D^+V(t, x)$ by

$$D^+V(t, x) = \lim_{h \to 0^+} \sup(\frac{1}{h})[V(t + h, x + hf(t, x)) - V(t, x)].$$

Consider the comparison differential system

$$\begin{cases} \frac{du}{dt} = g(t, u), & t \neq t_k, \\ \Delta u = B_k(u(t_k)), & t = t_k, k = 1, 2, \cdots, \\ u(t_0^+) = u_0. \end{cases} \quad (2)$$

Where $g \in PC[R_+ \times G, R^n], B_k \in C[G, R^n]$, and G is a cone in R^n. Let $S(\rho) = \{u \in G : \|u\| < \rho, \rho > 0\}, w \in PC[R_+ \times S(\rho), G]$. And for $(t, u) \in R_+ \times S(\rho), h > 0$, define the function $D^+w(t, x)$ by

$$D^+w(t, u) = \lim_{h \to 0^+} \sup(\frac{1}{h})[w(t + h, u + hg(t, u)) - w(t, u)].$$

Definition 4 *The trivial solution $x = 0$ of (1) is equistable, if for each $\varepsilon > 0, t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$, which is continuous in t_0 for each ε, such that the inequality $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon, t \geq t_0$.*

Other stability notions can be similarly defined (see [2]).

Definition 5 *The trivial solution $u = 0$ of (2) is φ_0-equistable, if given $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon)$ continuous in t_0 for each ε, such that the inequality $(\varphi_0, u_0) < \delta$ implies $(\varphi_0, r(t)) < \varepsilon, t \geq t_0$,where $\varphi_0 \in G_0^*$.*

Note In definition 5,and for the rest of this paper, $r(t)$ denotes the maximal solution of (2) relative to the cone $G \subset R^n$.

Other φ_0-stability concepts can be similarly defined.

Definition 6 *A function $a(\cdot)$ is said to belong to the class K, if $a \in C[(0, \rho), R_+], a(0) = 0$ and $a(r)$ is strictly increasing in r.*

Definition 7 *(a) A function $w(t, u)$ is said to be positive definite relative to the cone G (or φ_0-positive definite)if there exists $a \in K$, such that*

$$a[(\varphi_0, r(t))] \leq (\varphi_0, w(t, u)), \varphi_0 \in G_0^*.$$

(b) A function $w(t,u)$ is said to be decrescent relative to the cone G (or φ_0-decrescent)if there exists $b \in K, \varphi_0 \in G_0^*$, such that

$$(\varphi_0, w(t,u)) \leq b[(\varphi_0, r(t))].$$

2. Main results

Theorem 1 *Assume that*

(i) $w \in PC[R_+ \times S(\rho), G], w(t,0) = 0, w(t,u)$ is locally Lipschitzian in u relative to G, and for each $(t,u) \in R_+ \times S(\rho), D^+ w(t,u) \leq_G 0$;

(ii) $g \in PC[R_+ \times G, R^n], g(t,0) = 0$;

(iii) For some $\varphi_0 \in G_0^*$ and $(t,u) \in R_+ \times S(\rho)$,

$$a[(\varphi_0, r(t)] \leq (\varphi_0, w(t,u)), a \in K;$$

(iv) $(\varphi_0, w(t_i + 0, u + B_i(u))) \leq (\varphi_0, w(t_i, u)), B_i(0) = 0, i = 1, 2, \cdots$,
then the trivial solution $u = 0$ of (2) is φ_0-equistable.

Proof Since $w(t,0) = 0$ and $w(t,u_0)$ is continuous in t_0, then given $a_1(\varepsilon) > 0, t_0 \in R_+$, there exists δ_1, such that $\|u_0\| < \delta_1$,implies $\|w(t_0^+, u_0)\| < a_1(\varepsilon), a_1 \in K$.

Now for some $\varphi_0 \in G_0^*, \|\varphi_0\| \cdot \|u_0\| < \|\varphi_0\|\delta_1$ implies $\|\varphi_0\| \cdot \|w(t_0^+, u_0)\| < \|\varphi_0\| a_1(\varepsilon)$. Thus

$$|(\varphi_0, u_0)| \leq \|\varphi_0\| \cdot \|u_0\| < \|\varphi_0\|\delta_1,$$

implies

$$|(\varphi_0, w(t_0^+, u_0))| \leq \|\varphi_0\| \cdot \|w(t_0^+)\| < \|\varphi_0\| a_1(\varepsilon).$$

It follows that

$$(\varphi_0, u_0) < \delta \Longrightarrow (\varphi_0, w(t_0^+, u_0)) < a(\varepsilon),$$

where $\|\varphi_0\|\delta_1 = \delta, \|\varphi_0\| a_1(\varepsilon) = a(\varepsilon), a \in K$.

Let $u(t)$ be any solution of (2) such that $(\varphi_0, u_0) < \delta$,then by (i) w is nonincreasing and so $w(t,u) \leq w(t_0^+, u_0), t \geq t_0$. Thus $(\varphi_0, u_0) < \delta$, implies

$$a[(\varphi_0, r(t))] \leq (\varphi_0, w(t,u)) \leq (\varphi_0, w(t_0^+, u_0)) < a(\varepsilon) \Rightarrow (\varphi_0, r(t)) \leq \varepsilon, t \geq t_0.$$

Theorem 2 *Let the condition (i),(ii) and (iv) of theorem 1 hold.Assume further that for some $\varphi_0 \in G_0^*, (t,u) \in R_+ \times S(\rho), a[(\varphi_0, r(t))] \leq (\varphi_0, w(t,u)) \leq b[(\varphi_0, r(t))], a, b \in K$.*

Then the trivial solution $u = 0$ of (2) is uniformly φ_0-stable.

Proof For $\varepsilon > 0$, let $\delta = b^{-1}[a(\varepsilon)]$ independent of t_0 for $a, b \in K$. Let $u(t)$ be any solution of (2) such that $(\varphi_0, u_0) < \delta$. Then w is nonincreasing and so

$$(\varphi_0, w(t,u)) \leq (\varphi_0, w(t_0^+, u_0)).$$

Thus

$$a[(\varphi_0, r(t))] \leq (\varphi_0, w(t,u)) \leq (\varphi_0, w(t_0^+, u_0)) \leq b[(\varphi_0, u_0)] < b(\delta) < a(\varepsilon).$$

So $(\varphi_0, u_0) \leq \delta$ implies $(\varphi_0, w(t,u)) < \varepsilon$. □

Theorem 3 *Let the conditions of Theorem 1 hold with $D^+w(t,u) \leq_G 0$ replaced by*

$$D^+(\varphi_0, w(t,u)) \leq -c[(\varphi_0, w(t,u))], c \in K, \tag{3}$$

then the solution $u = 0$ of (2) is equi-asymptotically φ_0-stable.

Proof By Theorem 1, the trivial solution of (2) is φ_0-equistable. By (3), $w(t,u)$ is monotone decreasing and hence the limit $w^* = \lim_{t\to\infty} w(t,u)$ exists. We claim that $w^* = 0$.

Suppose $w^* \neq 0$.Then $c(w^*) \neq 0, c \in K$.Since $c(r)$ is monotone, $c[(\varphi_0, w(t,u))] > c[(\varphi_0, w^*)]$, and so $D^+(\varphi_0, w(t,u)) \leq -c[(\varphi_0, w^*)]$.

Integrating we obtain

$$(\varphi_0, w(t,u)) \leq -c[(\varphi_0, w^*)](t - t_0) + (\varphi_0, w(t_0^+, u_0)).$$

Thus as $t \to \infty$ and for some $\varphi_0 \in G_0^*$,we have $(\varphi_0, w(t,u)) \to -\infty$. This contradicts the condition $a[(\varphi_0, r(t))] \leq (\varphi_0, w(t,u))$. It follows that $w^* = 0$.

So $(\varphi_0, w(t,u)) \to 0$ as $t \to \infty$ and $(\varphi_0, r(t)) \to 0$ as $t \to \infty$.

Thus given $\varepsilon > 0, t_0 \in R_+$,there exist $\delta = \delta(t_0)$ and $T = T(t_0, \varepsilon)$ such that for $t \geq t_0 + T, (\varphi_0, u_0) < \delta$ implies $(\varphi_0, r(t)) < \varepsilon$. □

Theorem 4 *Assume that*

(i) $w \in PC[R_+ \times S(\rho), G], w(t,0) = 0$, and $w(t,u)$is locally Lipschitzian in u relative to the cone K for $t \in R_+$;

(ii) For each $t \in R_+, (t,u) \in R_+ \times S(\rho)$, and $c \in K$

$$D^+(\varphi_0, w(t,u)) \leq -c[(\varphi_0, r(t))].$$

(iii) $a[(\varphi_0, r(t))] \leq (\varphi_0, w(t,u)) \leq b[(\varphi_0, r(t))], a, b \in K$;

(iv) $(\varphi_0, w(t_i + 0, u + B_i(u))) \leq (\varphi_0, w(t_i, u)), B_i(0) = 0, i = 1, 2, \cdots$,

then the trivial solution $u = 0$ of (2) is uniformly asyptotically φ_0- stable.

Proof Let $\varepsilon > 0$ be given. Choose $\delta = \delta(\varepsilon)$ independent of t_0. Let $u(t)$ be a solution of (2) such that $(\varphi_0, u_0) < \delta$. Then by Theorem 2,$u = 0$ is uniformly φ_0-stable. Let

$$w^* = \{\sup(\varphi_0, w(t_0^+, u_0)) : (\varphi_0, u_0) < \delta\}.$$

Set $T(\varepsilon) = \frac{w^*}{c(\varepsilon)}, c \in K$, then

$$(\varphi_0, r(t)) < \varepsilon, (\varphi_0, u_0) < \delta, t \geq t_0 + T(\varepsilon). \tag{4}$$

Suppose (4) is not true, then there would exist at least one $t \geq t_0 + T(\varepsilon)$ such that $(\varphi_0, u_0) < \delta$ implise $(\varphi_0, r(t)) \geq \varepsilon$.

Since $c \in K$,from condition (ii), $D^+(\varphi_0, w(t,u)) \leq -c(\varepsilon)$.

Integrating, we obtain

$$(\varphi_0, w(t,u)) \leq (\varphi_0, w(t_0^+, u_0)) - c(\varepsilon)(t - t_0).$$

For $t \geq t_0 + T(\varepsilon)$ and sufficiently large t,this contradicts (iii), so case (4) is established. □

Theorem 5 *Assume that*

(i) $V \in PC[R_+ \times S_\rho, G], V(t,x)$ *is locally Lipschitzian in* x *relative to* G, *and for* $(t,x) \in R_+ \times S_\rho, D^+V(t,x) \leq_G g(t, V(t,x))$;

(ii) $g \in PC[R_+ \times G, R^n]$ *and* $g(t,u)$ *is quasimonotonely increasing in* u *relative to* G *for each* $t \in R_+$;

(iii) $f(t,0) = 0, g(t,0) = 0$, *for some* $\varphi_0 \in G_0^*, (t,x) \in R_+ \times S_\rho$,

$$b(\|x\|) \leq (\varphi_0, V(t,x)) \leq a[(t, \|x\|)], a, b \in K;$$

(iv) $B_i \in C[G, R^n], i = 1, 2, \cdots, \psi_i(u) = u + B_i(u)$ *are monotonely increasing in* G *and* $B_i(0) = 0, I_i(0) = 0$;

(v) $(\varphi_0, V(t_i + 0, x + I_i(x))) \leq (\varphi_0, \psi_i(w(t_i, x))), i = 1, 2, \cdots,$

then the trivial solution $x = 0$ *of (1) satisfies each one of the stability notions of Definitiom 4, if the trivial solution* $u = 0$ *of (2) satisfies the corresponding one of the stability notions of Definition 5.*

Proof (a) Let $0 < \varepsilon < \rho$ and $t \in R_+$, suppose that the trivial solution $u = 0$ of (2) is φ_0-equistable. Then given $b(\varepsilon) > 0, t_0 \in R_+$, there exists $\delta = \delta(t_0, \varepsilon)$, such that $(\varphi_0, u_0) < \delta$ implies $(\varphi_0, r(t)) < b(\varepsilon), t \geq t_0$.

Choose $a[(t_0^+, \|x\|)] = (\varphi_0, u_0)$, then

$$(\varphi_0, V(t_0^+, x_0)) \leq a[(t_0^+, \|x\|)] = (\varphi_0, u_0) \rightarrow V(t_0^+, x_0) \leq_G u_0.$$

Let $x(t; t_0, x_0)$ be any solution of (1) such that $V(t_0^+, x_0) \leq_G u_0$, then $V(t,x) \leq_G r(t)$.

Now choose $\delta_1 > 0$ such that $a[(t_0, \delta_1)] = \delta$. Thus the inequalities $\|x_0\| < \delta_1$ and $a[(t_0^+, \|x_0\|)] < \delta$ hold simultaneously .

Thus

$$b(\|x\|) \leq (\varphi_0, V(t,x)) \leq (\varphi_0, r(t)) < b(\varepsilon) \Rightarrow \|x(t; t_0, x_0)\| < \varepsilon,$$

whenever $\|x_0\| < \delta_1$.

(b) In the proof of (a) choose $\delta = \delta(\varepsilon)$ independent of t_0 and follow the same argument as in (a) to obtain the result.

(c) Suppose that the trivial solution $u = 0$ of (2) is quasi-equiasymptotically φ_0-stable, then following the same arguments as in (a), for all $t \geq t_0 + T(\varepsilon)$, we find that there exists a positive function $\delta = \delta(t_0, \varepsilon)$ satisfying the inequalities $\|x_0\| < \delta$ and $a[(t_0^+, \|x_0\|)] < \delta_0$ simultaneously, it then follows that

$$\|x(t; t_0, x_0)\| < \varepsilon, \|x_0\| < \delta_0, t \geq t_0 + T.$$

If this was not true,there would exist a divergent sequence $\{t_k\}, t_k \geq t_0 + T$, and a solution $x(t; t_0, x_0)$ of (1) such that whenever $x_0 < \delta$,we have that $\|x(t; t_0, x_0)\| = \varepsilon$.

Using Theorem 3.1 in [3] we are led to a contradiction:

$$b(\varepsilon) \leq (\varphi_0, V(t_k, x(t_k; t_0, x_0))) \leq (\varphi_0, r(t_k; t_0, u_0)) < b(\varepsilon).$$

(d) Since (a) and (c) are verified together, then $x = 0$ is equiasymptotically φ_0-stable.

(e) Since (b) holds, choose δ_0 and T in (c) independent of t_0 and proceed as in (c) to obtain the result. □

Theorem 6 *Let condition (i), (ii), (iv), (v) of Theorem 5 hold. Assume further that for $c > 0, d > 0, (\varphi_0, u_0) \le \|x_0\|^d$ and $c\|x\|^d \le (\varphi_0, V(t,x))$. If the trivial solution $u = 0$ of (2) is exponentially asymptotically φ_0- stable, then the trivial solution $x = 0$ of (1) is exponentially asymptotically stable.*

Proof Let $x(t; t_0, x_0)$ be any solution of (1), then we have that $V(t,x) \le_G r(t)$. Thus $c\|x\|^d \le (\varphi_0, V(t,x)) \le (\varphi_0, r(t))$.

Since the trivial solution $u = 0$ of (2) is exponentially asymptotically φ_0-stable, then there exist $\sigma > 0, \alpha > 0$ which are both real numbers such that

$$(\varphi_0, r(t)) \le \sigma(\varphi_0, u_0)\exp[-\alpha(t - t_0)], t \ge t_0,$$

and

$$c\|x\|^d \le \sigma(\varphi_0, u_0)\exp[-\alpha(t - t_0)].$$

This implies that

$$\|x\| \le M\|x_0\|\exp[-\beta(t - t_0)], t \ge t_0, \frac{\sigma}{c} = M, \frac{\alpha}{d} = \beta.$$

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脉冲比较微分系统解的 φ_0- 稳定性

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摘 要: 本文主要利用分段连续的 Lyapunov 函数得到脉冲比较微分系统 (2) 的 φ_0- 稳定性，并且通过比较方程，得到脉冲微分系统 (1) 的稳定性.

关键词: 脉冲微分系统； φ_0- 稳定；一致 φ_0- 稳定；等度渐近 φ_0- 稳定；一致渐近 φ_0- 稳定.