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On the Sign Pattern Matrices with Nonpositive k-power *

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Abstract: A matrix whose entries are $+$, $-$, and 0 is called a sign pattern matrix. Let k be arbitrary positive integer. We first characterize sign patterns A such that $A^k \leq 0$. Further, we determine the maximum number of negative entries that can occur in A whenever $A^k \leq 0$. Finally, we give a necessity and sufficiency condition for $A^2 \leq 0$.

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1. Introducation

A *sign pattern matrix* is a matrix whose entries are in the set $\{+,-,0\}$. Associated with each n by n sign pattern matrix $A=(a_{ij})$ is a class of real matrices, called the *sign pattern class* of A, defined by $Q(A)=\{B\in M_n(R)\mid \text{sign } b_{ij}=a_{ij} \text{ for all } i \text{ and } j\}$.

The set of all n by n sign pattern matrices is denoted by Q_n. Let $A=(a_{ij})\in Q_n$. Then $D(A)$, the directed graph (digraph) of A, is the digraph with vertex set $V(D(A))=\{1,2,\cdots,n\}$ and arc set $E(D(A))=\{(i,j)\mid a_{ij}\neq 0\}$. If $a_{ij}=+[-]$, then we say the arc (i,j) is positive [negative], we denote this arc by $i\xrightarrow{+}j$ $[i\xrightarrow{-}j]$. By a *walk* W in $D(A)$, we mean a sequence of arcs (i_0,i_1), (i_1,i_2), $\cdots$, (i_{k-1},i_k). The length of W, denoted by $l(W)$, is k. The above walk W can also be represented as $(i_0,i_1,i_2,\cdots,i_k)$. We say W is positive [negative], and we write $\text{sign}(W)=+[\text{sign}(W)=-]$, if W contains an even [odd] number of negative arcs. W is a *closed walk* if $i_0=i_k$. A walk $P=(i_0,i_1,i_2,\cdots,i_k)$ is a *path* if $i_0,i_1,i_2,\cdots,i_k$ are distinct. P is a *cycle* if $i_0=i_k$ (sometimes this is referred to as a simple cycle). A cycle [walk] with length k is a k-cycle [k-walk]. Note that a diagonal entry is a 1-cycle. If k is even [odd], we say the k-cycle, or k-walk, is even [odd]. If a cycle or walk contains only negative [positive] arcs, we say it is entrywise negative [entrywise positive].

The notions defined above for $D(A)$ can be modified slightly and defined for $A=(a_{ij})$. Thus a *walk* W of A (or in A) is a formal product of the form $W=a_{i_0i_1}a_{i_1,i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}$,

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where the entries involved are nonzero. Paths, closed walks and simple cycles in A are defined analogously, as well as their lengths and signs.

In [1], authors consider sign patterns whose square is nonpositive. The goal of this paper is to investigate the n by n ($n \geq 2$) sign pattern matrices A such that $A^k \leq 0$ for any positive integer k. Of course, this implies that A^k is defined.

Let $A \in Q_n$. For any positive integer k, A^k is defined, that is, exists as a pattern, if the (i,j) entry of A^k is unambiguously defined for all $1 \leq i,j \leq n$, and we write $A^k \in Q_n$. If we denote the (i,j) entry of A^k by $(A^k)_{ij}$, then it is clear that $(A^k)_{ij}$ is unambiguously defined iff no two walks of length k from i to j have opposite signs. If A^k exists as a pattern, then $(A^k)_{ij} = \sum \mathrm{sign}(W)$, where W runs over the set of k-walks of A from i to j. Of course, the sum is zero if there is no such walk. Thus the following basic theorem is clear and we omit the proof of it.

Theorem 1.1 *Let $A \in Q_n$ and k be arbitrary positive integer. Then $A^k \leq 0$ if and only if every k-walk of A is negative.*

Corollary 1.2 *Let $A \in Q_n$ be irreducible. If $A^k \leq 0$, then all nonzero entries in rth row (rth column, respectively) of A have same sign for $r = 1, 2, \cdots, n$.*

In Section 2 and 3 of this paper, we characterize these sign patterns with k is even and odd, respectively, and determine the maximum number of negative entries in these sign patterns. In final section, we consider the special case $k = 2$. We then obtain a necessity and sufficiency condition of $A^2 \leq 0$.

In order to simplify our notation, in the remainder of this paper, we let the index set $\{1, 2, \cdots, n\}$ be represented by N. We let $N_-(A)$ denote the number of negative entries in A, and N_n^k denote the maximum number of negative entries that can occur in A whenever $A^k \leq 0$.

2. Patterns with nonpositive $2s$-power

In this section, we consider the n by n sign pattern matrices A such that $A^{2s} \leq 0$, where s is an arbitrary positive integer. By Theorem 1.1, the first four results are clear, and we may omit the proofs of them.

Theorem 2.1 *If $A \in Q_n$ with $A^{2s} \leq 0$, then every cycle of A is not entrywise negative or entrywise positive.*

Corollary 2.2 *If $A \in Q_n$ with $A^{2s} \leq 0$, then $a_{ii} = 0$ for all i.*

Corollary 2.3 *Let $A \in Q_n$ with $A^{2s} \leq 0$. Then $a_{ij}a_{ji} \neq +$ for any i and j.*

Theorem 2.4 *Let $A \in Q_n$ with $A^{2s} \leq 0$. Then A has no l-cycles for any $l \mid s$.*

Theorem 2.5 *Let $A \in Q_n$ with $A^{2s} \leq 0$. Then A has no odd cycles.*

Proof Suppose γ is an odd cycle of A and $l(\gamma) = l$. Let $W = \gamma^{2s}$. Then $\mathrm{sign}(W) = [\mathrm{sign}(\gamma)]^{2s} = +$. By Theorem 1.1, $\mathrm{sign}(W) = (-)^l = -$, yielding a contradiction. Thus A has no odd cycles. □

Corollary 2.6 *Let $A \in Q_n$ be irreducible such that $A^{2s} \leq 0$. Then A is permutation similar to a block matrix of the form*

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \tag{1}$$

where the diagonal blocks are square.

Proof Since $n \geq 2$ and A is irreducible, $D(A)$ is strongly connected. It is well known that a strongly connected digraph with no odd cycles is bipartite. Then A is permutation similar to the desired form. □

We now turn our attention to finding the maximum number of negative entries in A, whenever $A \in Q_n$ such that $A^{2s} \leq 0$.

Lemma 2.7 *Let $A \in Q_n$ be irreducible such that $A^{2s} \leq 0$. Then $N_-(A) \leq [\frac{n^2}{4}]$, and equality may hold whenever s is odd.*

Proof By Corollary 2.6, A is permutation similar to a block matrix of the form given (1), where the diagonal blocks are m by m and $(n-m)$ by $(n-m)$ zero matrices, respectively. Then $N_-(A) = N_-(B) + N_-(C)$. Since $a_{ij}a_{ji} \neq +$ for all i and j, $N_-(A) \leq m(n-m) \leq [\frac{n}{2}][\frac{n+1}{2}] = [\frac{n^2}{4}]$.

When s is odd, we let

$$A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where $B > 0$, $C < 0$, and the diagonal blocks are zero matrixs of order $[\frac{n}{2}]$ and $[\frac{n+1}{2}]$, respectively. Then $A^{2s} \leq 0$ and $N_-(A) = [\frac{n^2}{4}]$. □

Lemma 2.8 *Let $A \in Q_n$ be reducible such that $A^{2s} \leq 0$, and let $r = [\frac{n}{2s}]$, $p = n - 2sr$. Then $N_-(A) \leq \frac{1}{2}[n^2 - p(r+1)^2 - (2s-p)r^2]$, and equality may hold.*

Proof Let $A \in Q_n$ be reducible such that $A^{2s} \leq 0$. Assume that A^- is the sign pattern matrix obtained from A by replacing the positive entries of A by 0, and that $D(A^-)$ is the digraph of A^-. It is clear that $V(D(A^-)) = V(D(A)) = N$, $N_-(A) = |E(D(A^-))|$. By Theorem 2.1, $D(A^-)$ has no cycles. We now let $V_1 = \{i \in N :$ there is no arc (j,i) in $D(A^-)$ for any $j \in N\ \}$. Then V_1 is nonvacuous. For $m = 2, 3, \cdots$, we let $V_m = \{j \in N :$ there is a $(m-1)$-path of $D(A^-)$ from i to j for some $i \in V_1$, but for any $i \in V_1$ and $l > m-1$, there is no l-path from i to j in $D(A^-)\ \}$. It is clear that V_i $(i = 1, 2, \cdots)$ is a independent set of $D(A^-)$ if it is nonvacuous. By Theorem 1.1, there is no $2s$-path in $D(A^-)$, $V_j = \emptyset$ for $j > 2s$, and $N = V_1 \cup V_2 \cup \cdots \cup V_{2s}$. Assume $|V_i| = n_i$ for $i = 1, 2, \cdots, 2s$. Then $N_-(A) \leq \sum_{1 \leq i < j \leq 2s} n_i n_j = \frac{1}{2}(n^2 - \sum_{i=1}^{2s} n_i^2)$.

If $2s \geq n$, then $N_-(A) \leq \frac{1}{2}(n^2 - n)$. If $2s < n$, noticing $n = 2sr + p$, then

$$N_-(A) \leq \frac{1}{2}[n^2 - p(r+1)^2 - (2s-p)r^2].$$

On the other hand, we consider the n by n reducible sign pattern matrix as follows

$$A = \begin{pmatrix} 0 & A_{12} & A_{13} & \cdots & A_{1m} \\ 0 & 0 & A_{23} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & A_{m-1,m} \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

where $A_{ij} < 0$ for $1 \le i < j \le m$, and the diagonal blocks are n_i by n_i, $i = 1, 2, \cdots, m$, zero matrices, respectively.

(1) If $2s \ge n$, we let $m = n$, and $n_1 = n_2 = \cdots = n_n = 1$. It is clear that $A^{2s} = 0$, and $N_-(A) = \frac{1}{2}(n^2 - n)$.

(2) If $2s < n$, we let $m = 2s$, and $n_1 = \cdots = n_p = r + 1$, $n_{p+1} = \cdots = n_{2s} = r$. It is clear that $A^{2s} = 0$, and $N_-(A) = \frac{1}{2}[n^2 - p(r+1)^2 - (2s - p)r^2]$.

The proof of the theorem is completed. □

Combining Lemmas 2.7 and 2.8, we have the following theorem.

Theorem 2.9 *Let s be a positive integer. Then*

$$N_n^{2s} = \frac{1}{2}[n^2 - p(r+1)^2 - (2s - p)r^2],$$

where $r = [\frac{n}{2s}]$ and $p = n - 2sr$.

3. Patterns with nonpositive $(2s+1)$-power

In this section, we consider the n by n sign pattern matrices A such that $A^{2s+1} \le 0$, where s is arbitrary positive integer. By Theorem 1.1, the first two theorems are clear, and we state them without proofs.

Theorem 3.1 *Let $A = (a_{ij}) \in Q_n$ with $A^{2s+1} \le 0$. If $a_{ii} \ne 0$ for some i, then $a_{ii} = -$, and every walk of $D(A)$ which contains the vertex i is entrywise negative.*

Theorem 3.2 *Let $A \in Q_n$ with $A^{2s+1} \le 0$. Then every l-cycle is entrywise negative for any $l \mid 2s$.*

Theorem 3.3 *Let $A \in Q_n$ with $A^{2s+1} \le 0$. If γ is a l-cycle of A, then $\mathrm{sign}(\gamma) = (-)^l$.*

Proof Suppose $W = \gamma^{2s+1}$. Then W is a closed walk of A of length $l(2s+1)$, and $\mathrm{sign}(W) = [\mathrm{sign}(\gamma)]^{2s+1} = \mathrm{sign}(\gamma)$. On the other hand, by Theorem 1.1, $\mathrm{sign}(W) = (-)^l$. Thus $\mathrm{sign}(\gamma) = (-)^l$. □

Theorem 3.4 *For arbitrary positive integer s, $N_n^{2s+1} = n^2$.*

Proof Let $A \in Q_n$ be entrywise negative. It is clear that $A^{2s+1} < 0$, and $N_-(A) = n^2$. □

Theorem 3.5 *Let $A \in Q_n$ such that $A^{2s+1} \le 0$. If A have m irreducible components, then $N_-(A) \le n^2 - \frac{(2n-m)(m-1)}{2}$, and equality may hold.*

Proof By Theorem 3.4, we may assume $2 \le m \le n$. Then A is permutation similar to a block matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{mm} \end{pmatrix}, \tag{2}$$

where each $A_{ii} \in Q_{n_i}$ is irreducible, and $\sum_{i=1}^{m} n_i = n$. Thus $N_-(A) \le \frac{1}{2}(n^2 + \sum_{i=1}^{m} n_i^2)$. Since $n_i \ge 1$ for $1 \le i \le m$, $\sum_{i=1}^{m} n_i^2 \le m-1+(n-m+1)^2$, and $N_-(A) \le n^2 - \frac{(2n-m)(m-1)}{2}$.

We now let A have the block upper triangular form given in (2), where each A_{ij} is entrywise negative for $1 \le i \le j \le m$, $A_{ii} \in Q_{n_i}$ and $n_1 = \cdots = n_{m-1} = 1$, $n_m = n-m+1$. It is clear that $A^{2s+1} \le 0$, and $N_-(A) = n^2 - \frac{(2n-m)(m-1)}{2}$. □

4. A special case

In this section, we consider the special case $k = 2$. We given a necessity and sufficiency condition of $A^2 \le 0$. We also characterize the n by n sign patterns A such that $N_-(A) = N_n^2 = [\frac{n^2}{4}]$.

Theorem 4.1 *Let $A \in Q_n$. Then $A^2 \le 0$ if and only if A is permutation similar to a block matrix of the form*

$$\begin{pmatrix} 0 & A_{12} & A_{13} & 0 \\ A_{21} & 0 & A_{23} & 0 \\ 0 & 0 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & 0 \end{pmatrix},$$

where $A_{12} \ge 0$, $A_{13} \ge 0$, $A_{42} \ge 0$, $A_{21} \le 0$, $A_{23} \le 0$, $A_{41} \le 0$, and the diagonal blocks are square or empty.

Proof Since sufficiency is clear, we only prove necessity.

Assume $A \in Q_n$ such that $A^2 \le 0$. Let $I^-(A)$ $(T^-(A))$ denote the vertex subset of $D(A)$ such that for each vertex $i \in I^-(A)$ $(i \in T^-(A))$, there exist some vertex j with $i \xrightarrow{-} j$ $(j \xrightarrow{-} i)$ in $D(A)$. Similarly, $I^+(A)$ $(T^+(A))$ the vertex subset of $D(A)$ such that for each vertex $i \in I^+(A)$ $(i \in T^+(A))$, there exist some vertex j with $i \xrightarrow{+} j$ $(j \xrightarrow{+} i)$ in $D(A)$. Since $A^2 \le 0$, by Theorem 1.1, $I^-(A) \cap T^-(A) = \emptyset$ and $I^+(A) \cap T^+(A) = \emptyset$.

Let $X = T^-(A) \cap I^+(A)$, $Y = I^-(A) \cap T^+(A)$, $Z = (T^+(A)\backslash Y) \cup (T^-(A)\backslash X)$ and $W = V\backslash(X \cup Y \cup Z)$. Since every 2-path in $D(A)$ is negative, by the definitions of above, we have the results as follows.

(1) The some of X, Y, Z and W can be vacuous, but $X \cup Y \cup Z \cup W = V$.

(2) Each one of X, Y, Z and W is a independent set of $D(A)$ if it is nonvacuous.

(3) Each arc from a vertex in X to a vertex in $Y \cup Z$ is positive; each arc from a vertex in Y to a vertex in $X \cup Z$ is negative.

(4) There is no arc whose terminal is in W, but there may be arc whose initial is in W.

(5) If there exists a arc whose initial is in W, then the arc is positive if its terminal is in Y, negative if its terminal is in X. But if its terminal is in Z, then the arc can be

positive or negative.

Combining (1)–(5), we obtain the theorem. □

Theorem 4.2 *If $A \in Q_n$ such that $A^2 \le 0$, then $N_-(A) = N_n^2 = [\frac{n^2}{4}]$ if and only if A is permutation similar to a block matrix of the form*

$$\pm \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where $B \ge 0$, $C < 0$, and the diagonal blocks are zero matrices of order $[\frac{n}{2}]$ and $[\frac{n+1}{2}]$, respectively.

Proof We use the marks on the proof of Theorem 4.1.

Now assume

$$A = \pm \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where $B \ge 0$, $C < 0$, and the diagonal blocks are zero matrices of order $[\frac{n}{2}]$ and $[\frac{n+1}{2}]$, respectively. Then it is not difficult to verify that $A^2 \le 0$ and $N_-(A) = [\frac{n^2}{4}]$.

Conversely, assume that A have $[\frac{n^2}{4}]$ negative entries. Since $I^-(A) \cap T^-(A) = \emptyset$, letting $|I^-(A)| = k$, then $N_-(A) \le |I^-(A)| \cdot |T^-(A)| \le k(n-k)$, and we conclude that $k(n-k) = [\frac{n^2}{4}]$. Thus $k = [\frac{n}{2}]$ or $[\frac{n+1}{2}]$ from the above. It follows that A is permutation similar to the desired form. □

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k 次幂非正的符号模式矩阵

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摘 要: 本文首先对使得 $A^k \le 0$ 的符号模式矩阵 A 进行了刻画 (k 为任意正整数)，进而决定了这类矩阵中负元个数的最大值. 最后给出了使得 $A^2 \le 0$ 的符号模式矩阵 A 的充分必要条件.

关键词: 符号模式；矩阵；有向图.