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The Extension of Infinitesimal Prolongation Theorem and Its Application *

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Abstract: Infinitesimal prolongation theorem is extended from sequences to nets based on k-saturated nonstandard model. As its an application, a main property about topology of uniform convergence is proved. The proof is much simpler than it was, meanwhile the nonstandard characteristics of convergence with respect to u.c. topology is given.

Key words: Infinitesimal Prolongation Theorem; k-saturation; monad of uniformity; topology of uniform convergence.

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The directed set D, topological spaces X and Y, which are refered to in this paper, are all included in the set S of individuals of the superstructure U. $^*U$ is k-saturated nonstandard model of U, where k is a cardinal number and $k >$ card(U) (the cardinal number of U). We denote f for $^*f$, the extension of f on $^*U$.

Infinitesimal Prolongation Theorem *If* $\{s_n : n \in {}^*N\}$ *is internal and* $s_n \approx 0$ *for finite* n, *there is infinite* H *such that* $s_n \approx 0$ *for all* $n \leq H$, *where* N *and* $^*N$ *denote natural number set and hypernatural number set,respectively.*

Infinitesimal Prolongation Theorem is a very important theorem about internal sequences. In this paper, we first prove that Infinitesimal Prolongation Theorem still hold on nets of $^*R$ with k-saturated nonstandard model. Then an important application of this extension will be given (Maybe A. Robinson did not extend the theorem from sequences to nets because he did not find the extension of k-saturated model).

Definition 1 *Let* k *be an infinite cardinal number. The ultrapower nonstandard model* $^*U$ *is* k*-saturated if* $\{A_i : i \in I\}$ *is a collection of internal sets in* $^*U$ *having f.i.p. (finite intersection property) and card* $(I) < k$, *then* $\cap\{A_i : i \in I\} \neq \Phi$.

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Let $(D, \leq)$ be a directed set of U, by Transfer Principle, $({}^*D, \leq)$ is a directed set of ${}^*U$. The following theorem is the main result of this paper.

Theorm 1 *Let $\{s_n : n \in {}^*D\}$ be an internal net in ${}^*R$ and for each $n \in D, s_n \approx 0$, then there is an infinite element $q \in {}^*D$ such that $s_q \approx 0$.*

Proof For each $m \in N$, where N is the natrual number set, let $A_m = \{n \in {}^*D : |s_n| < 1/m\}$. By internality theorem, A_m is internal subset of ${}^*D$. Obviously, $D \subset A_m$ for each $m \in N$.

For each element (m,p) of $N \times D$, let $B(m,p) = \{n \in {}^*D : n \in A_m \& n \geq p\}$.

Since D is directed and $D \subset A_m, B(m,p)$ is non-empty. Because A_m is internal, $B(m,p)$ is an internal subset of ${}^*D$ by Internality Therorem.It is easily to see that the internal collection $\{B(m,p) : (m,p) \in N \times D\}$ have finite intersection property. Hence, by k-saturation of ${}^*U(k > \text{card}(U) \geq \text{card}(N \times D))$.

$$\cap\{B(m,p) : (m,p) \in N \times D\} \neq \Phi.$$

Taking $q \in \cap\{B(m,p) : (m,p) \in N \times D\}$. Then by definitions of $B(m,p)$ and A_m, q is an infinite element of ${}^*D$(i.e.$q \geq p$ for each $p \in D$) and $s_q \approx 0$.

As an application of Theorem 1, we will prove a very important property about topology of uniform convergence, that is, the family of continuous functions is a close subspace of functional space with respect to topology of uniform convergence.

Definition 2 *Let $(Y, \mathcal{V})$ be an uniform space. We call $\cap\{{}^*V : V \in \mathcal{V}\}$ the monad of the uniform structure $\mathcal{V}$ which is denoted by $M(\mathcal{V})$.*

It is easy to verify that for each $y \in Y$, the monad of y with respect to topology of uniform is $M(y) = M(\mathcal{V})[y] = \{z \in {}^*Y : (y,z) \in M(\mathcal{V})\}$.

Definition 3 *Let F be the family of functions from a set X to an uniform space $(Y, \mathcal{V})$. For each $V \in \mathcal{V}$, let $W(V) = \{(f,g) \in F \times F$: for each $x \in X, (f(x), g(x)) \in V\}$. The uniform structure $\mathcal{U}$ based on the collection $\{W(V) : V \in \mathcal{V}\}$ is called uniform structure of uniform convergence which is denoted by u.c. The topology of the uniform structure $\mathcal{U}$ is called topology of uniform convergence,denoted by topology of u.c.*

Lemma 1 *Let F be defined as definition 3 and $\{f_n : n \in D\}$ be a net in F. $\{f_n : n \in D\}$ converges to f with respect to topology of u.c.if and only if for each infinite element $n \in {}^*D$ and for each $x \in X$,*

$$f_n(x) \in M(\mathcal{V})[f(x)].$$

Proof First suppose that the net $\{f_n : n \in D\}$ converges to f with respect to the topology of u.c. Then for every element V of $\mathcal{V}$, there exists $p \in D$ such that

$$(\forall n \in D)(n \geq p \rightarrow f_n \in W(V)[f])$$

holds. By Transform Principle,

$$(\forall n \in {}^*D)(n \geq p \rightarrow f_n \in {}^*W(V)[f])$$

holds. Hence for each infinite $n \in {}^*D, f_n \in {}^*W(V)$ for each $V \in \mathcal{V}$. Appling Transform Principle again, we have

$$ {}^*W(V) = \{(f,g) \in {}^*F \times {}^*F : \text{for } \forall x \in {}^*X, (f(x), g(x)) \in {}^*V\}. $$

So, $f_n \in {}^*W(V)$ implies $f_n(x) \in {}^*V[f(x)]$. By V's arbitrariness, for each infinite $n \in {}^*D$ and every $x \in X$,

$$ f_n(x) \in \cap\{{}^*V[f(x)] : V \in \mathcal{V}\} = M(\mathcal{V})[f(x)]. $$

Next suppose that for each infinite element $n \in {}^*D$ and every $x \in {}^*X, f_n(x) \in M(\mathcal{V})[f(x)]$. Then for every element V of $\mathcal{V}$

$$ (\exists p \in {}^*D)(\forall n \in {}^*D)(\forall x \in {}^*X)(n \geq p \to f_n(x) \in {}^*V[f(x)]) $$

holds (take p to be an infinite element). Transform Principle implies

$$ (\exists p \in D)(\forall n \in D)(\forall x \in X)(n \geq p \to f_n(x) \in V[f(x)]). $$

By the definition of $W(V), f_n \in W(V)[f]$ whenever $n \geq p$. Hence the net $\{f_n \in D\}$ converges to f with respect to the topology of u.c..

Note If $\mathbf{P}$ is a normal family of uniform structure $\mathcal{V}$, then the collection $\{V_{d,r} : d \in P, r \in R^+\}$ is a subbasis of $\mathcal{V}$. Therefore,

$$ M(\mathcal{V}) = \cap\{{}^*V_{d,r} : d \in P, r \in R^+\} = \{(y,z) \in {}^*Y \times {}^*Y : \text{for } \forall d \in \mathbf{P}, d(y,z) \approx 0\} $$
$$ (\text{where } V_{d,r} = \{(y,z) \in Y \times Y : d(y,z) < r\}). $$

Hence Lemma 1 can be expressed as following: The net $\{f_n : n \in D\}$ converges to f with respect to the topology of u.c.if and only if for each infinite element $n \in {}^*D$ and every pseudo-metric $d \in \mathbf{P}, d(f_n(x), f(x)) \approx 0$ for every $x \in {}^*X$.

Lemma 2[1] *Suppose that f is a function from topological space X to topological space Y.f is continuous if and only if for each $x \in X$,*

$$ f[M(x)] \subset M[f(x)]. $$

Following, we apply Theorem 1 and Lemma 1,2 to prove a very important property about topology of uniform convergence.

Theorem 3 *Let F be the collection of continuous functions from topological space X to uniform space $(Y, \mathcal{V})$. Then F is a closed subspace of functional space Y^X with respect to toptology of u.c.*

Proof It is sufficient to prove that every limit function of convergent nets of F belongs to F.

Suppose that $\{f_n : n \in D\}$ is a net of F and converges to f with respect to topology of u.c., by Lemma 1, for each element $d \in \mathbf{P}$, where $\mathbf{P}$ is normal family of uniform structure $\mathcal{V}$, and every infinite element $n \in {}^*D, d(f_n(x), f(x)) \approx 0$ for each $x \in {}^*X$.

Since $f_n \in F$ whenever $n \in D$, by Lemma 2, whenever $x \in M(p)$, where p is an element of $X, d(f_n(x), f_n(p)) \approx 0$ for each $n \in D$. Hence there exists an infinite element $q \in {}^*D$ such that $d(f_q(x), f_q(p)) \approx 0$, by Theorem 1. Therefore, whenever $x \in M(p)$ we have

$$d(f(x), f(p)) \leq d(f_q(x), f(x)) + d(f_q(x), f_q(p)) + d(f_q(p), f(p)) \approx 0.$$

By the arbitrariness of d, it is hold that $x \in M(p)$ implies that

$$f(x) \in M(\mathcal{V})[f(p)].$$

Hence f is continuous, that is, $f \in F$ by Lemma 2.

Obviously the proof is much simpler than it was. Theorem 1 is key of the simplicity.

References:

[1] DAVIS M. *Applied Nonstandard Anslysis* [M]. Wiley, New York, 1977.

[2] CUTLAND N J. *Nonstandard measure theory and its applications* [J]. Bull. London Math. Soc., 1983, **15**: 529–589.

[3] LUXEMBURG W A J. *A General Theory of Monads, Applications of Model Theory to Algebra and Porbability* [M]. Holt. Rinehart and Winston, New York, 1969, 18-86.

[4] KELLY J L. *General Topology* [M]. Van Nostrand, Springer, 1955.

无穷小延伸定理的推广及其应用

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摘 要: 本文在饱和的非标准模型中把无穷小延伸定理推广到网的情形，做为应用，利用推广了的无穷小延伸定理给出拓扑空间中连续泛函的一个重要性质的离散化证明.

关键词: 无穷小延伸定理；饱和；一致结构的单子；一致收敛拓扑.