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Geometric-Harmonic Mean and Characterizations of Some Mean-Values *

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Abstract: The purpose of this paper is to provide a direct proof on the fact that the geometric-harmonic mean of any two positive numbers can be calculated by a first complete elliptical integral, and then to give new characterizations of some mean-values.

Key words: Arithmetic mean; geometric mean; harmonic mean; geometric-harmonic mean; first complete elliptic integral.

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1. Geometric-Harmoinc mean

Let a and b be any two positive real numbers. The arithmetic, geometric and harmonic mean-values of a and b are denoted by $A = A(a,b)$, $G = G(a,b)$ and $H = H(a,b)$, respectively. It is well known that the arithmetic-geometric mean-value of a and b is defined by $AG(a,b) = \lim a_n = \lim b_n$, where $a_0 = a$, $b_o = b$ and $a_{n+1} = A(a_n,b_n)$, $b_{n+1} = G(a_n,b_n)$. In [1], we see that $AG(a,b)$ with $0 < a \leq b$ can be calculated by a first complete elliptic integral as follows:

$$AG(a,b) = \pi / \int_0^{\pi} (a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^{-1/2} \, \mathrm{d}\varphi.$$

Also, the arithmetic-harmonic mean-value of a and b is defined by $AH(a,b) = \lim a_n = \lim b_n$, where $a_0 = a$, $b_0 = b$ and $a_{n+1} = A(a_n,b_n)$, $b_{n+1} = H(a_n,b_n)$. It is easy to prove that $AH(a,b) = (ab)^{1/2}$.

Now we would like to consider the geometric-harmonic mean-value of any two positive numbers a and b. Let $a_0 = a$, $b_0 = b$ and $a_{n+1} = G(a_n,b_n)$, $b_{n+1} = H(a_n,b_n)$. It is easy to prove that both $\lim a_n$ and $\lim b_n$ exist and are equal. We define the common value as the geometric-harmonic mean of a and b and denote it by $GH(a,b)$. We will show that $GH(a,b)$ can also be calculated by a first complete elliptic integral.

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Theorem 1 *If* $0 < a \le b$, *then*

$$GH(a,b) = (1/\pi)\int_0^{\pi}(\cos^2\varphi/a^2 + \sin^2\varphi/b^2)^{-1/2}\,\mathrm{d}\varphi.$$

Proof For $0 < a \le b$, consider the integral

$$L = \int_0^{\pi/2}(\cos^2\varphi/a^2 + \sin^2\varphi/b^2)^{-1/2}\,\mathrm{d}\varphi.$$

Let $\sin\varphi = (2/a)\sin\theta/((1/a+1/b)+(1/a-1/b)\sin^2\theta)$. After some calculation, we obtain

$$(\cos^2\varphi/a^2 + \sin^2\varphi/b^2)^{-1/2}\,\mathrm{d}\varphi = (((a+b)/2ab)^2\cos^2\theta + (1/ab)\sin^2\theta)^{-1/2}\,\mathrm{d}\theta.$$

Set $a_1 = 2ab/(a+b)$ and $b_1 = (ab)^{1/2}$. We have

$$L = \int_0^{\pi/2}(\cos^2\theta/a_1^2 + \sin^2\theta/b_1^2)^{-1/2}\,\mathrm{d}\theta.$$

By repeating this procedure, we obtain

$$L = \int_0^{\pi/2}(\cos^2\varphi/a_n^2 + \sin^2\varphi/b_n^2)^{-1/2}\,\mathrm{d}\varphi \quad (n = 1,2,3,\ldots),$$

where $a_0 = a$, $b_0 = b$ and $a_n = H(a_{n-1}, b_{n-1})$, $b_n = G(a_{n-1}, b_{n-1})$.

Clearly, $(\pi/2)a_n < L < (\pi/2)b_n$, and it follows $L = (\pi/2)GH(a,b)$. Hence

$$GH(a,b) = (2/\pi)L = (1/\pi)\int_0^{\pi}(\cos^2\varphi/a^2 + \sin^2\varphi/b^2)^{-1/2}\,\mathrm{d}\varphi.$$

2. Characterizations of some mean-values

In [2] we have seen the following three mean-values of a and b:

(1) $M(a,b;p(r)) = p^{-1}((1/2\pi)\int_0^{2\pi} p(r)\,\mathrm{d}\theta)$, where $p: R^+ \to R$, $p''(x)$ is a continuous function in R^+, $p = p(x)$ is strictly monotonic in R^+, and denote $(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}$ by r.

(2) $M(a,b;q(s)) = q^{-1}((1/2\pi)\int_0^{2\pi} q(s)\,\mathrm{d}\theta)$, where $q: R^+ \to R$, $q''(x)$ is a continuous function in R^+, $q = q(x)$ is strictly monotonic in R^+, and denote $a\sin^2\theta + b\cos^2\theta$ by s.

(3) $M(a,b;u(t)) = u^{-1}((1/2\pi)\int_0^{2\pi} u(t)\,\mathrm{d}\theta)$, where $u: R^+ \to R$, $u''(x)$ is a continuous function in R^+, $u = u(x)$ is strictly monotonic in R^+, and denote $(\sin^2\theta/a + \cos^2\theta/b)^{-1}$ by t.

Now we would like to consider the following mean-value of a and b:

(4) $M(a,b;v(z)) = v^{-1}((1/2\pi)\int_0^{2\pi} v(z)\,\mathrm{d}\theta)$, where $v: R^+ \to R$, $v''(x)$ is a continuous function in R^+, $v = v(x)$ is strictly monotonic in R^+, and denote $(\cos^2\theta/a^2 + \sin^2\theta/b^2)^{-1/2}$ by z.

We have obtained several new characterizations of some mean-values.

Lemma 1[2] *If a and b are positive real constants, then we have*

(1) $(1/2\pi)\int_0^{2\pi}(\cos^2\theta/a^2+\sin^2\theta/b^2)^{-1}\,\mathrm{d}\theta = ab;$
(2) $(1/2\pi)\int_0^{2\pi}(\cos^2\theta/a^2+\sin^2\theta/b^2)^{-2}\,\mathrm{d}\theta = ab(a^2+b^2)/2;$
(3) $(1/2\pi)\int_0^{2\pi}\log(\cos^2\theta/a^2+\sin^2\theta/b^2)\,\mathrm{d}\theta = 2\log((a+b)/2ab).$

Lemma 2 *Let $v: R^+\to R$, assume that $v''(x)$ is continuous in R^+. If we set*

$$f(a,b) = (1/2\pi)\int_0^{2\pi} v(z)\,\mathrm{d}\theta = (1/2\pi)\int_0^{2\pi} v((\cos^2\theta/a^2+\sin^2\theta/b^2)^{-1/2})\,\mathrm{d}\theta$$

for all positive a and b, then

$$f_{aa}(c,c) = (3/8)v''(c) - (3/8c)v'(c),$$

where c is an arbitrarily fixed real number.

Proof The proof follows from differentiation under the integral sign.

Theorem 2 *Let $A(\neq 0)$ and B be arbitrary real constants.*

(1) $M(a,b;v(z)) = GH(a,b)$ holds for all positive real numbers a and b if and only if $v(z) = Az+B$.

(2) $M(a,b;v(z)) = G(a,b)$ holds for all positive real numbers a and b if and only if $v(z) = Az^2+B$.

(3) $M(a,b;v(z)) = H(a,b)$ holds for all positive real numbers a and b if and only if $v(z) = A\log z + B$.

(4) $M(a,b;v(z)) = (H(a^2,b^2))^{1/2}$ holds for all positive real numbers a and b if and only if $v(z) = A(1/z^2)+B$.

(5) There exists no $v(z)$ such that $M(a,b;v(z)) = A(a,b)$ holds for all positive real numbers a and b.

Proof (1) Suppose $M(a,b;v(z)) = GH(a,b)$. Then

$$(1/2\pi)\int_0^{2\pi} v(z)\,\mathrm{d}\theta = v((1/\pi)\int_0^{\pi}(\cos^2\theta/a^2+\sin^2\theta/b^2)^{-1/2}\,\mathrm{d}\theta).$$

Operating on both sides of the above equality with $\partial^2/\partial a^2$ and setting $a=c$, $b=c$ in the resulting equality, by Lemma 2 we obtain

$$(3/8)v''(c) - (3/8c)v'(c) = (1/4)v''(c) - (3/8c)v'(c),$$

and it follows $v''(c) = 0$. Since c is an arbitrarily fixed positive real number, we can replace c by a positive real variable z in the above equality. Hence we have $v''(z) = 0$ in R^+. This implies

$$v(z) = Az+B$$

in R^+, where A and B are real constants with $A\neq 0$.

On the contrary, suppose $v(z) = Az+B$ and $z = (\cos^2\theta/a^2+\sin^2\theta/b^2)^{-1/2}$. Then $v^{-1}(z) = (z-B)/A$, and so

$$\begin{aligned} M(a,b;v(z)) &= v^{-1}((1/2\pi)\int_0^{2\pi} v(z)\,\mathrm{d}\theta) = (1/2\pi)\int_0^{2\pi}(\cos^2\theta/a^2+\sin^2\theta/b^2)^{-1/2}\,\mathrm{d}\theta \\ &= GH(a,b). \end{aligned}$$

Clearly, we can also prove (2), (3) and (4) by the same arguments and using Lemma 1 and Lemma 2.

(5) Suppose there exists some $v(z)$ such that $M(a,b;v(z)) = A(a,b)$ holds for all positive real numbers a and b. Then

$$(1/2\pi)\int_0^{2\pi} v(z)\,\mathrm{d}\theta = v((a+b)/2).$$

Operating on both sides of the above equality with $\partial^2/\partial a^2$, setting $a=c$, $b=c$, where c is an arbitrarily fixed positive real number, in the resulting equality, using Lemma 2, we can get

$$v''(c) - (3/c)v'(c) = 0.$$

Replacing c by a positive real variable z in the above equality yields $v''(z)-(3/z)v'(z)=0$ in R^+. This implies

$$v(z) = Az^4 + B$$

in R^+, where A and B are real constants with $A \neq 0$.

However, by using $v^{-1}(z) = ((z-B)/A)^{1/4}$, $z = (\cos^2\theta/a^2 + \sin^2\theta/b^2)^{-1/2}$, and Lemma 1(2), after some calculations we obtain

$$M(a,b;v(z)) = (ab(a^2+b^2)/2)^{1/4}$$

for all positive a and b. This leads to a contradiction and so (5) is proved.

Remark We can prove that $M(a,b;v(z)) = (ab(a^2+b^2)/2)^{1/4}$ holds for all positive real numbers a and b if and only if $v(z) = Az^4 + B$, where A and B are arbitrary real constants with $A \neq 0$.

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几何－调和平均和一些平均值的特征

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摘　要: 本文的目的是直接证明任何两个正数的几何－调和平均值都可以用第一类完全椭圆积分来计算，并且给出一些平均值新的特征.

关键词: 算术平均；几何平均；调和平均；几何－调和平均；第一类完全椭圆积分.