Journal of Mathematical Research & Exposition
Vol.23, No.2, 237–240, May, 2003

A Partial Approximation Shepard Method Based on Cardinal Spline About Messy Data *

NIE Hui, ZENG Long, LUO Xiao-nan
(Computer Application Institute, Zhongshan University, Guangzhou 510275, China)

Abstract: With regards to Shepard method, in the paper, we present a better one based on partial approximation to fit messy data. In the method, partial cubic cardinal spline function is chosen as weight function $\varphi(x)$ in the Shepard formula which is $\varphi(x)\in C^2$ and has good attenuation characteristics. So the traditional Shepard method is improved and the better results can be achieved in practical applications.

Key words: Shepard method; messy data fitting; partial approximation; cardinal spline

Classification: AMS(2000) 65D17/CLC number: O241.5

Document code: A **Article ID:** 1000-341X(2003)02-0237-04

1. Introduction

Shepard method is applied in the messy data fitting for surface. We can construct the surface by averaging the reciprocal of distance with weight. Generally, the surface can be expressed as follows.

Assume there are a serials of messy vertices (x_i, y_i) and the corresponding value of the function $f_i = f(x_i, y_i), i = 1, 2, ..., n$. Let $r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2}$, the fitting surface $z = f(x, y)$ can be expressed as

$$z = f(x,y) = \begin{cases} (\sum_{i=1}^{n} \frac{f_i}{r_i^{\mu}})/(\sum_{i=1}^{n} \frac{1}{r_i^{\mu}}), & r_i \neq 0, \\ f_i, & r_i = 0. \end{cases} \tag{1}$$

The formula is a globe interpolation expression about $(x_i, y_i), i = 1, 2, ..., n$. If (x, y) is not an interpolation vertex, then $f(x, y)$ is the average of all the value of function f_i with weight, and the weight $\frac{1}{r_i^{\mu}}$ is related to (x, y). μ is a positive real number as which we usually select a constant>1.

Let $\varphi(r_i) = \frac{1}{r_i^{\mu}}$, then (1) can be expressed as

$$z = f(x,y) = \begin{cases} \sum_{i=1}^{n} f_i(\varphi(r_i))^{\mu} / \sum_{i=1}^{n} (\varphi(r_i))^{\mu}, & r_i \neq 0, \\ f_i, & r_i = 0. \end{cases}$$

*Received date: 2000-10-18
Biography: NIE Hui (1971-), female, Ph.D.

In the Shepard method, it is important to choose a good weight $\varphi(r_i) \in C^2$, and the better decay characteristics of the chosen weight function, the better results for messy data fitting can be achieved.

2. A partial approximation Shepard method based on cardinal spline

2.1 Cubic cardinal spline function

Suppose $\varphi(x)$ is a cubic spline function with all integer numbers in $(-\infty, +\infty)$ as its nodes. $\varphi(x)$ satisfies the interpolation condition $\varphi(j) = \delta_{0j}(j = 0, \pm1, \pm2, ...)$ and is limitary in the whole number axis. So the function can be expressed as

$$\varphi(x) = \begin{cases} (3\lambda+2)x^3 - 3(\lambda+1)x^2 + 1, & 0 \le x \le 1, \\ 3\lambda^j[(\lambda+1)(x-j)^3 - (\lambda+2)(x-j)^2 + (x-j)], & j \le x \le j+1 \ (j = 1, 2, \cdots), \\ \varphi(-x), & x < 0 \end{cases}$$

with $\lambda = \sqrt{3} - 2 \approx -0.268$.

2.2 A partial approximation method based on cardinal spline

Suppose there are a serials of messy vertexes (x_i, y_i) and the corresponding value of the function $f_i = f(x_i, y_i), i = 1, 2, ..., n$. We construct the interpolation surface $f(x, y)$ as follows.

Step 1. Let $R > 0$,and make sure there are a certain number of control vertexes in the field of a circle with center (x_i, y_i) and radius R. Suppose m is the number of the vertexes being denoted $(x_j^i, y_j^i), j = 1, 2, ..., m$.

Step 2. Calculate the distance between (x_i, y_i) and $(x_j^i, y_j^i), j = 1, 2, ..., m$, that is

$$R_j = \sqrt{(x_j^i - x_i)^2 + (y_j^i - y_i)^2}, \quad j = 1, 2, ..., m.$$

Suppose that $R_j (j = 1, 2, ..., m)$ are not equal. We sort R_j with incremental degree. Let $0 < R_1 < R_2 < ... < R_m$.

Step 3. Using partial approximation method, we construct $\varphi(x)$. The cubic cardinal spline function can be expressed as

$$\varphi(x) = \begin{cases} (3\lambda+2)x^3 - 3(\lambda+1)x^2 + 1, & 0 \le x \le 1, \\ 3\lambda^{j-1}[(\lambda+1)(x-j+1)^3 - (\lambda+2)(x-j+1)^2 + \\ \quad (x-j+1)], & j-1 \le x < j \\ & (j = 1, 2, \cdots, m-1), \\ 3\lambda^j[(\lambda+1)(x-j)^3 - (\lambda+2)(x-j)^2 + (x-j)], & j \le x < j+1 (j = m-1), \\ 0, & x \ge m. \end{cases}$$

In order to make $\varphi(x) \in C^2$,we reconstruct $\varphi(x)$ in $m-1 \le x < m$. Because the conditions need to be satisfied for $\varphi(x) \in C^2$,we have

$$\varphi_+(m-1) = \varphi_-(m-1), \ \varphi_+'(m-1) = \varphi_-'(m-1), \ \varphi_+''(m-1) = \varphi_-''(m-1),$$
$$\varphi_-(m) = 0, \ \varphi_-'(m) = 0, \ \varphi_-''(m) = 0.$$

Using Hermite interpolation, we have

$$\varphi(x) = 3\lambda^{m-1}H_0(x) - 6\lambda^{m-1}(\lambda+2)F_0(x),\ m-1 \le x < m,$$
$$H_0(x) = -(x-m)^3[3(x-m+1)^2 + (x-m+1)],\ F_0(x) = -\frac{(x-m)^3(x-m+1)^2}{2}.$$

Step 4. Choose the uniform cubic cardinal spline function as weight $\varphi(r_i)$, and transform the field of definitions to make the interpolated vertexes and integer in the number axis bijection.

Let $r_i(x,y) = k+\frac{R-R_k}{R_{k+1}-R_k}, R_k \le R < R_{k+1}, k = 0,1,2,\cdots,m-1, R = \sqrt{(x-x_i)^2+(y-y}$

Step 5. Put $r_i(x,y)$ in the formula of cubic cardinal spline function $\varphi(r_i)$. We have

$$\varphi(r_i) = \begin{cases} (3\lambda+2)r_i^3 - 3(\lambda+1)r_i^2 + 1, & 0 \le r_i \le 1, \\ 3\lambda^{j-1}[(\lambda+1)(r_i-j+1)^3 - (\lambda+2)(r_i-j+1)^2 + \\ \quad (r_i-j+1)], & j-1 \le r_i < j \\ & (j = 1,2,...,m-1), \\ 3\lambda^{m-1}H_0(r_i) - 6\lambda^{m-1}(\lambda+2)F_0(r_i)], & m-1 \le r_i < m, \\ 0, & r_i \ge m, \end{cases}$$

$$H_0(r_i) = -(r_i-m)^3[3(r_i-m+1)^2 + (r_i-m+1)],$$
$$F_0(r_i) = -\frac{(r_i-m)^3(r_i-m+1)^2}{2}.$$

Step 6. By the five steps ahead, we reconstruct the partial approximation weight function based on cardinal spline $\varphi(r_i)$, and get the improved vector expression of Shepard surface.

$$\vec{P}(x,y) = \frac{\sum_{i=1}^N \vec{f_i}\varphi(r_i)^\mu}{\sum_{j=1}^N \varphi(r_j)^\mu}, \quad \vec{f_i} = (x_i, y_i, f_i).$$

3. Contrast of the partial approximation methods

Choose $R > 0$, we have for Shepard's partial approximation method have that

$$\varphi(r) = \begin{cases} 1/r, & 0 < r \le R/3, \\ \frac{27(r/R-1)^2}{4R}, & R/3 < r \le R, \\ 0, & r > R. \end{cases}$$

Clearly, $\varphi(r)$ is differentiable, and $\varphi(r) = 0$ when $r > R$. The method implements partial approximation, but $\varphi(r) \in C^1$, the smoothness of the fitting surface is not perfect. The improved Shepard method makes use of the cubic cardinal spline function, by transaction, $\varphi(r) \in C^2$,so the effect of fitting is better than that of fitting with weight function $\varphi(r) \in C^1$.

From the properties of cubic cardinal spline function, we know that $\varphi(x)$ is an attenuating wave motion function, which attenuate λ times pre segment. That is to say if only change the value of the function in the position of node i , the influence to the position after j segments will come to decrease rapidly at speed of λ^j. The property makes the spreading speed of errors attenuated rapidly, which makes the calculation more stably.

From the point of view of theoretical analysis, the method in the paper is easy to extend to the high dimensional space. As for n-dimensional space ($n > 3$), by bijection, any $n-1$ dimensional sphere or circle can mapped to the domain $[0, 1]$. After we have implemented the transformation of field of definitions, by making use of partial approximation cardinal spline, we can also construct the fitting surface in the high dimensional space.

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基于基样条局部逼近散乱数据拟合中的 Shepard 方法

聂 卉， 曾 龙， 罗笑南

(中山大学计算机应用研究所，广东 广州 510275)

摘 要: 本文针对散乱数据拟合的 Shepard 方法，提出了一种局部逼近的新方法. 该方法以局部三次基样条函数作为 Shepard 公式中的权函数，新的权函数具有良好的衰减性和二阶连续性，从而改进了传统方法的不足之处，使实际应用效果更好.

关键词: Shepard 方法；散乱数据拟合；局部逼近；基样条.