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# Global Attractivity in a Delay Difference Equation *

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**Abstract:** A new sufficient condition is given for the global attractivity of solutions of the delay difference equation $x_{n+1} = x_n f(x_n, x_{n-1})$, $n = 0, 1, \ldots$. As an application, our results partly confirm a conjecture of G. Ladas.

**Key words:** delay difference equation; global asymptotic stability; global attractivity; semicycle; periodic point of prime period two.

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## 1. Introduction

In recent years, there has been a lot of interest in the study of the global attractivity of delay difference equations. We refer to [1–5] and the references cited there for more details. In this paper, we first recall the following delay difference equation

$$x_{n+1} = \frac{a + bx_n}{A + x_{n-1}}, \qquad n = 0, 1, \cdots, \tag{1}$$

where

$$a,\ b,\ A \in (0, \infty) \tag{2}$$

and the initial values $x_{-1}, x_0$ are arbitrary positive numbers.

Eq.(1) has a unique positive equilibrium point $\bar{x}$. It is the unique positive root of the equation

$$\bar{x} = \frac{a + b\bar{x}}{A + \bar{x}}, \tag{3}$$

i.e.,

$$\bar{x} = (b - A + \sqrt{(b - A)^2 + 4a})/2. \tag{4}$$

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It can be easily shown that $\bar{x}$ is locally asymptotically stable. There are many papers dealing with the global asymptotic stability of $\bar{x}$. Some results are summed in [1]. See also [5]. For the sake of convenience, we present them as follows:

**Theorem G.L** *Suppose that (2) holds, and one of the following conditions is valid:*
(1) $b < A$;
(2) $b \geq A$ and $a \leq Ab$;
(3) $b \geq A$ and $Ab < a \leq A(2A - b)$;
(4) $b \geq ((1+\sqrt{5})/2)^{1/2}A$, $Ab < a$, and $b^2/A \leq \bar{x} \leq 2b$.
*Then $\bar{x}$ is globally asymptotically stable.*

Here, positive equilibrium point $\bar{x}$ of Eq.(1) is said to be globally asymptotically stable if it is both locally asymptotically stable and globally attractive. While, $\bar{x}$ is said to be globally attractive if any solution $\{x_n\}$ of Eq.(1) converges to $\bar{x}$ for arbitrary initial values $x_{-1}, x_0 \in (0,\infty)$. Clearly, positive equilibrium $\bar{x}$ of Eq.(1) is globally asymptotically stable if and only if $\bar{x}$ is globally attractive.

A natural question is whether (2) is sufficient for $\bar{x}$ to be globally asymptotically stable. For this problem, G. Ladas presented the following conjecture in both [1] and [2].

**Conjecture G.L** *Assume that (2) holds. Then every positive solution $\{x_n\}$ of Eq. (1) tends to a finite limit as $n \to \infty$.*

If the conjecture is true, then it is easy to obtain $\lim_{n\to\infty} x_n = \bar{x}$. As far as we know, the best results related to the conjecture are given in Theorem G.L.. Motivated by Conjecture G.L., we will next investigate the global attractivity of a more general delay difference equation than Eq.(1), namely,

$$x_{n+1} = x_n f(x_n, x_{n-1}), \qquad n = 0, 1, \ldots, \tag{5}$$

where the initial values $x_{-1}, x_0$ are arbitrary positive numbers.

Throughout this paper, we always suppose that the function $f(x,y)$ in Eq.(5) satisfies the following hypotheses:

(H1) $f(x,y) \in C((0,\infty)\times[0,\infty),(0,\infty))$ and $\lim_{x\to 0^+} xf(x,y)$ exists;

(H2) $f(x,y)$ is nonincreasing in $x, y$;

(H3) the equation $f(x,x) = 1$ has only one positive solution $\bar{x}$. (Here and in the sequel, we also always denote by $\bar{x}$ equilibrium point or fixed point of an equation.)

It is obvious under the assumptions (H1) and (H3) that Eq.(5) has a unique equilibrium $\bar{x}$ and every solution $\{x_n\}$ of Eq.(5) is positive. Eq.(5) has been investigated by V.L. Kocic and G. Ladas in [3]. They obtained the following result:

**Theorem V.G** *Suppose that the hypotheses (H1)–(H3) hold and that $xf(x,x)$ is increasing in $x$. Then the positive equilibrium point $\bar{x}$ of Eq.(5) is globally asymptotically stable.*

Our main aim in this paper is to replace the condition "$xf(x,x)$ is increasing in $x$" by another condition. Now define the functions $G(x,y)$ and $F(x)$ respectively as follows:

$$G(x,y) = yf(y,x)f(\bar{x},x), \tag{6}$$

$$F(x) = \max_{x \le y \le \bar{x}} G(x, y) \qquad \text{for} \quad 0 \le x \le \bar{x},$$
$$= \min_{\bar{x} \le y \le x} G(x, y) \qquad \text{for} \quad x > \bar{x}. \tag{7}$$

A positive semicycle of a nontrivial solution of Eq.(5) consists of a 'string' terms $\{x_r, x_{r+1}, \cdots, x_s\}$ with $x_i \ge \bar{x}$, $r \le i \le s$ and $x_{r-1} < \bar{x}$, $x_{s+1} < \bar{x}$. A negative semicycle of a nontrivial solution of Eq.(5) consists of a 'string' terms $\{x_p, x_{p+1}, \cdots, x_q\}$ with $x_i < \bar{x}$, $p \le i \le q$ and $x_{p-1} \ge \bar{x}$, $x_{q+1} \ge \bar{x}$.

For other concepts in this paper, see [3, 5].

## 2. Several Lemmas

**Lemma 1** *Suppose that hypotheses (H1)–(H3) hold. Then every positive solution of Eq.(5) is bounded and persists.*

**Lemma 2** *Suppose that hypotheses (H1)–(H3) hold. Let $\{x_n\}$ be a nontrivial positive solution of Eq.(5) such that for some $n_0 \ge 0$, either $x_n \ge \bar{x}$ for $n \ge n_0$ or $x_n \le \bar{x}$ for $n \ge n_0$. Then for $n \ge n_0 + 1$, the sequence $\{x_n\}$ is monotonic and $\lim_{n\to\infty} x_n = \bar{x}$.*

**Lemma 3** *Let $F(x) \in C([0, \infty), (0, \infty))$ be a nonincreasing function, and let $\bar{x}$ denote the unique fixed point of $F(x)$. Then the following statements are equivalent:*

*(a) $\bar{x}$ is the only fixed point of $F^2(x)$ in $(0, \infty)$, where $F^2(x) = F(F(x))$;*

*(b) If $\lambda$ and $\mu$ are positive numbers such that $F(\mu) \le \lambda \le \bar{x} \le \mu \le F(\lambda)$, then $\lambda = \mu = \bar{x}$.*

For the proofs of Lemma 1, 2 and 3, see Theorem 2.2.1, Lemma 2.2.1 and Lemma 1.6.3 in [3] respectively.

**Lemma 4** *Suppose that the hypotheses (H1)–(H3) hold. Let $\{x_n\}$ be a strict oscillatory solution of Eq.(5). Then the extreme of any semicycle, except perhaps for the first semicycle, must occur at either the first or second term.*

**Proof** Let $\{x_n\}$ be strictly oscillatory about $\bar{x}$, $\{x_{r_i+1}, x_{r_i+2}, \cdots, x_{s_i}\}$ be the $i$-th negative semicycle of $\{x_n\}$ which is followed by the $i$-th positive semicycle $\{x_{s_i+1}, x_{s_i+2}, \cdots, x_{t_i}\}$, and $x_{m_i}$ and $x_{M_i}$ be the extreme values in the two semicycle with the smallest possible indexes $m_i$ and $M_i$, respectively. Then we claim that $m_i - r_i \le 2$ and $M_i - s_i \le 2$. In fact, if $m_i - r_i > 2$, then $x_{m_i-1} < \bar{x}$, $x_{m_i-2} < \bar{x}$. So,

$$x_{m_i} = x_{m_i-1} f(x_{m_i-1}, x_{m_i-2}) \ge x_{m_i-1} f(\bar{x}, \bar{x}) = x_{m_i-1}.$$

This contradicts the assumption that $x_{m_i}$ is the smallest value in the semicycle. Analogously, we can show that $M_i - s_i \le 2$.

**Lemma 5** *Let $F(x)$ be defined by (7). Then $F(x) \in C([0, \infty), (0, \infty))$ and $F$ is nonincreasing in $[0, \infty)$.*

**Proof** The proof is completely similar to that of [3, Lemma 2.3.1 ] and so is omitted here.

## 3. Main Results

**Theorem 1** *Suppose that the hypotheses (H1)–(H3) hold and that the function $F$ defined by (7) has no periodic points of prime period 2. Then the positive equilibrium point $\bar{x}$ of Eq.(5) is globally attractive.*

**Proof** Let $\{x_n\}$ be an arbitrary positive solution of Eq.(5). If $\{x_n\}$ is not strictly oscillatory about $\bar{x}$, then according to Lemma 2, we have $\lim_{n\to\infty} x_n = \bar{x}$. Therefore, in order to establish the global attractivity of $\bar{x}$, it suffices in the following for us to treat only the case that $\{x_n\}$ is strictly oscillatory about $\bar{x}$.

Let $\{x_n\}$ be strictly oscillatory about $\bar{x}$, $\{x_{r_i+1}, x_{r_i+2}, \cdots, x_{s_i}\}$ be the i-th negative semicycle of $\{x_n\}$ which is followed by the i-th positive semicycle $\{x_{s_i+1}, x_{s_i+2}, \cdots, x_{t_i}\}$, and $x_{m_i}$ and $x_{M_i}$ be the extreme values in these two semicycles with the smallest possible indexes $m_i$ and $M_i$, respectively. Then it follows from Lemma 4 that

$$m_i - r_i \le 2, \qquad M_i - s_i \le 2. \tag{8}$$

By Lemma 1 we see that

$$\lambda = \lim_{n\to\infty} \inf x_n = \lim_{i\to\infty} \inf x_{m_i} \quad \text{and} \quad \mu = \lim_{n\to\infty} \sup x_n = \lim_{i\to\infty} \sup x_{M_i} \tag{9}$$

exist and are finite. So, there exist two positive constants $P$ and $Q$ with $0 < P \le Q < \infty$ such that

$$0 < P < \lambda \le \bar{x} \le \mu \le Q. \tag{10}$$

To complete the proof of the theorem, it suffices to show that

$$\lambda = \bar{x} = \mu. \tag{11}$$

For any given $\varepsilon \in (0, \lambda)$, from (9) we know that there exists an $n_0 \in N = \{1, 2, \cdots\}$ such that $\lambda - \varepsilon < x_n < \mu + \varepsilon$, for $n \ge n_0$. Let the index $i$ be so large that $r_i - 1 \ge n_0$. Based on (8), we only need to consider the following two cases.

Case (1): $m_i - r_i < 2$. Then we have

$$\begin{aligned} x_{m_i} &= x_{r_i+1} = x_{r_i} f(x_{r_i}, x_{r_i-1}) \ge x_{r_i} f(x_{r_i}, \mu+\varepsilon) \ge x_{r_i} f(x_{r_i}, \mu+\varepsilon) f(\bar{x}, \mu+\varepsilon) \\ &= G(\mu+\varepsilon, x_{r_i}) \ge \min_{\bar{x}\le y\le \mu+\varepsilon} G(\mu+\varepsilon, y) = F(\mu+\varepsilon). \end{aligned} \tag{12}$$

Case (2): $m_i - r_i = 2$. Then one can also see

$$\begin{aligned} x_{m_i} &= x_{r_i+2} = x_{r_i+1} f(x_{r_i+1}, x_{r_i}) \\ &= x_{r_i} f(x_{r_i}, x_{r_i-1}) f(x_{r_i+1}, x_{r_i}) \ge x_{r_i} f(x_{r_i}, \mu+\varepsilon) f(\bar{x}, \mu+\varepsilon) \\ &= G(\mu+\varepsilon, x_{r_i}) \ge \min_{\bar{x}\le y\le \mu+\varepsilon} G(\mu+\varepsilon, y) = F(\mu+\varepsilon). \end{aligned} \tag{13}$$

Therefore, from (12) and (13) we find $x_{m_i} \ge F(\mu+\varepsilon)$. Letting $i \to \infty$ and $\varepsilon \to 0^+$ in this inequality gives $\lambda \ge F(\mu)$. Analogously, we can derive $\mu \le F(\lambda)$. Combining these two inequalities and (10) we have $F(\mu) \le \lambda \le \bar{x} \le \mu \le F(\lambda)$. By the assumption that the function $F$ has no periodic points of prime period 2 and Lemma 3, we see that (11) is true and the proof is completed.

As an application, we now look at Eq.(1) again. Obviously, Eq.(1) can be rewritten in the form of Eq.(5) with

$$f(x,y)=(b+a/x)/(A+y), \qquad (x,y)\in(0,\infty)\times[0,\infty). \tag{14}$$

We have the following result.

**Theorem 2** *Suppose that (2) holds and that $a\le A(2A-b)$. Then the positive equilibrium point $\bar{x}$ of Eq.(1) is globally attractive.*

**Proof** We may easily test and verify that $f(x,y)$ as defined by (14) satisfies hypotheses (H1)–(H3). From (6) and (7), it is easy to see that

$$G(x,y)=\frac{(a+by)(b+\frac{a}{\bar{x}})}{(A+x)^2}, \quad F(x)=\frac{(a+b\bar{x})(b+\frac{a}{\bar{x}})}{(A+x)^2}=\frac{\bar{x}(A+\bar{x})^2}{(A+x)^2}.$$

In view of Theorem 1, it suffices to prove that $\bar{x}$ is the only fixed point of $F^2(x)$. This can be easily argued by the known assumption $a\le A(2A-b)$ and the expression of the function $F$. Thus, the proof is over.

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# 时滞差分方程的吸引性


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**摘 要**: 本文给出了时滞差分方程 $x_{n+1}=x_nf(x_n,x_{n-1}), n=0,1,\cdots$ 解的全局吸引性的一个新的充分条件；作为应用，部分解决了 G.Ladas 的一个猜想.



**关键词**: 时滞差分方程；全局渐近稳定性；全局吸引性；半环；素周期为 2 的周期点.