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一类 Fisher 方程的行波解*

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摘　要:本文讨论了一类广义 Fisher 方程,得到了它的多个显式行波解.

关键词:广义 Fisher 方程；行波解；显式行波解.
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1　引　言

考虑广义 Fisher 方程

$$u_t = Du_{xx} + Pu(1-u^\alpha)(q+u^\alpha),$$

其中 $D,P,\alpha>0,q\in R$. 它的行波解 $u=u(z)=u(x+ct)$,其中 $c>0$ 是波速；这类方程多次出现在生化反应的研究文献中,是一类重要的生化模型. 它的行波解对应着一个平面常微分系统奇点间的连接轨线,因此行波解的结果可直接应用于相平面上常微系统的全局分析及分支现象研究. 在一些文献中对这个方程的行波解已有讨论(参见[1],[2],[3]),但由于参数的复杂多变性,还有许多问题有待深入讨论.

对上面的方程作无量纲变换,即可得到如下类型的方程

$$u_t = u_{xx} + u(1-u^\alpha)(u^\alpha+q), \tag{1}$$

其中 $\alpha>0,q\in R$. 本文主要讨论(1) 的显式行波解,这些解可以用来定量描述方程的性质.

2　待定系数法

下面采用待定系数法来讨论(1) 的显式行波解. 将 $u=u(z)=u(x+ct)$ 代入(1) 式可得到

$$\ddot{u} - c\dot{u} + qu + (1-q)u^{\alpha+1} - u^{2\alpha+1} = 0.$$

令 $\dot{u}=p$,则 $\ddot{u}=p\dfrac{\mathrm{d}p}{\mathrm{d}u}$,代入上式有 $p\dfrac{\mathrm{d}p}{\mathrm{d}u} - cp + qu + (1-q)u^{\alpha+1} - u^{2\alpha+1} = 0$,考虑如

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下形式的解 $p = A_1 u + A_2 u^{\alpha+1}$，代入可得

$(A_1 u + A_2 u^{\alpha+1})[A_1 + A_2(\alpha+1)u^{\alpha}] - c(A_1 u + A_2 u^{\alpha+1}) + qu + (1-q)u^{\alpha+1} - u^{2\alpha+1} = 0.$

比较对应项的系数可得

$$\begin{cases} A_1^2 - cA_1 + q = 0, & (2) \\ (\alpha+2)A_1A_2 - cA_2 + (1-q) = 0, & (3) \\ (\alpha+1)A_2^2 - 1 = 0. & (4) \end{cases}$$

由(4)式可得 $A_2 = \pm \dfrac{\sqrt{\alpha+1}}{\alpha+1}$，由(3)式可得 $A_1 = \dfrac{c}{\alpha+2} + \dfrac{q-1}{(\alpha+2)A_2}$，由(2)式可得 $A_1 = \dfrac{c \pm \sqrt{c^2-4q}}{2}$，下面分几种情形进行讨论.

(1) $A_2 = \dfrac{\sqrt{\alpha+1}}{\alpha+1}$，$A_1 = \dfrac{c+\sqrt{c^2-4q}}{2}$. 此时有 $\dfrac{c+\sqrt{c^2-4q}}{2} = \dfrac{c}{\alpha+2} + \dfrac{q-1}{\alpha+2}\sqrt{\alpha+1}$，从中解得当 $q > 1$ 时，$c = \dfrac{\sqrt{\alpha+1}}{\alpha+1}(q+\alpha+1)$. 经计算当 $q > \alpha+1$ 时，才有 $A_1 = \dfrac{c+\sqrt{c^2-4q}}{2} = \dfrac{c}{\alpha+2} + \dfrac{q-1}{\alpha+2}\sqrt{\alpha+1} = \dfrac{\sqrt{\alpha+1}}{\alpha+1}q$ 成立. 此时有

$$\frac{\mathrm{d}u}{\mathrm{d}z} - \frac{\sqrt{\alpha+1}}{\alpha+1}qu = \frac{\sqrt{\alpha+1}}{\alpha+1}u^{\alpha+1}, \quad u^{\alpha} = \frac{-q}{1+e^{-\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha qz}}.$$

(2) $A_2 = \dfrac{\sqrt{\alpha+1}}{\alpha+1}$，$A_1 = \dfrac{c-\sqrt{c^2-4q}}{2}$. 此时有 $\dfrac{c-\sqrt{c^2-4q}}{2} = \dfrac{c}{\alpha+2} + \dfrac{q-1}{\alpha+2}\sqrt{\alpha+1}$，从而

$$c = \begin{cases} \dfrac{\sqrt{\alpha+1}}{\alpha+1}(q+\alpha+1)，当 q > -(\alpha+1) 时, \\ -\dfrac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1]，当 q < -\dfrac{1}{\alpha+1} 时. \end{cases}$$

1) $c = \dfrac{\sqrt{\alpha+1}}{\alpha+1}(q+\alpha+1)$ 时，经计算当 $-(\alpha+1) < q \leqslant \alpha+1$ 时，有 $A_1 = \dfrac{c-\sqrt{c^2-4q}}{2} = \dfrac{c}{\alpha+2} + \dfrac{q-1}{\alpha+2}\sqrt{\alpha+1} = \dfrac{\sqrt{\alpha+1}}{\alpha+1}q$ 成立，此时的解与情形(1)中相同.

2) $c = -\dfrac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1]$ 时，经计算当 $q < -\dfrac{1}{\alpha+1}$ 时，有 $A_1 = \dfrac{c-\sqrt{c^2-4q}}{2} = \dfrac{c}{\alpha+2} + \dfrac{q-1}{\alpha+2}\sqrt{\alpha+1} = -\dfrac{\sqrt{\alpha+1}}{\alpha+1}$ 成立，此时可以得到

$$\frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\sqrt{\alpha+1}}{\alpha+1}u = \frac{\sqrt{\alpha+1}}{\alpha+1}u^{\alpha+1}, \quad u^{\alpha} = \frac{1}{1+e^{\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha z}}.$$

(3) $A_2 = -\dfrac{\sqrt{\alpha+1}}{\alpha+1}$，$A_1 = \dfrac{c+\sqrt{c^2-4q}}{2}$. 此时有 $\dfrac{c+\sqrt{c^2-4q}}{2} = \dfrac{c}{\alpha+2} + \dfrac{1-q}{\alpha+2}\sqrt{\alpha+1}$，从中解得

$$c=\begin{cases}\dfrac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1], & q>-\dfrac{1}{(\alpha+1)}\text{ 时}, \\ -\dfrac{\sqrt{\alpha+1}}{\alpha+1}[q+(\alpha+1)], & q<-(\alpha+1)\text{ 时}.\end{cases}$$

1) $c=\frac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1)]$ 时，经计算当 $-\frac{1}{\alpha+1}<q<\frac{1}{\alpha+1}$ 时，有 $A_1=\frac{c+\sqrt{c^2-4q}}{2}=\frac{c}{\alpha+2}+\frac{1-q}{\alpha+2}\sqrt{\alpha+1}=\frac{\sqrt{\alpha+1}}{\alpha+1}$ 成立，此时可以得到

$$\frac{\mathrm{d}u}{\mathrm{d}z}-\frac{\sqrt{\alpha+1}}{\alpha+1}u=-\frac{\sqrt{\alpha+1}}{\alpha+1}u^{\alpha+1},\quad u^{\alpha}=\frac{1}{1+e^{-\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha z}}.$$

2) $c=-\frac{\sqrt{\alpha+1}}{\alpha+1}[q+(\alpha+1)]$ 时，经计算当 $q<-(\alpha+1)$ 时，有 $A_1=\frac{c+\sqrt{c^2-4q}}{2}=\frac{c}{\alpha+2}+\frac{1-q}{\alpha+2}\sqrt{\alpha+1}=-\frac{\sqrt{\alpha+1}}{\alpha+1}q$ 成立，此时可以得到

$$\frac{\mathrm{d}u}{\mathrm{d}z}+\frac{\sqrt{\alpha+1}}{\alpha+1}qu=-\frac{\sqrt{\alpha+1}}{\alpha+1}u^{\alpha+1},\quad u^{\alpha}=\frac{-q}{1+e^{\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha qz}}.$$

(4) $A_2=-\frac{\sqrt{\alpha+1}}{\alpha+1}$，$A_1=\frac{c-\sqrt{c^2-4q}}{2}$. 此时有 $\frac{c-\sqrt{c^2-4q}}{2}=\frac{c}{\alpha+2}+\frac{1-q}{\alpha+2}\sqrt{\alpha+1}$，

解得

$$c=\begin{cases}\dfrac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1)], & q>-\dfrac{1}{(\alpha+1)}\text{ 时}, \\ -\dfrac{\sqrt{\alpha+1}}{\alpha+1}[q+(\alpha+1)], & q<-(\alpha+1)\text{ 时}.\end{cases}$$

1) $c=\frac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1]$ 时，经计算当 $q\geqslant\frac{1}{\alpha+1}$ 时，有 $A_1=\frac{c-\sqrt{c^2-4q}}{2}=\frac{c}{\alpha+2}+\frac{1-q}{\alpha+2}\sqrt{\alpha+1}=\frac{\sqrt{\alpha+1}}{\alpha+1}$ 成立，此时的解与情形(3)中1)的解相同.

2) 当 $c=-\frac{\sqrt{\alpha+1}}{\alpha+1}[q+(\alpha+1)]$，$q<-(\alpha+1)$ 时无解.

3 显式行波解

综合以上的讨论结果，就可以得到方程(1)的一些显式行波解. 下面也分几种情况.

定理1 若 $-(\alpha+1)<q<0$，$c=\frac{\sqrt{\alpha+1}}{\alpha+1}(q+\alpha+1)$，$u^{\alpha}=\frac{-q}{1+e^{-\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha qz}}$，对于任意的 $\alpha>0$，方程(1)都有行波解

$$u(z)=\left(\frac{-q}{1+e^{-\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha qz}}\right)^{\frac{1}{\alpha}},\quad u(-\infty)=(-q)^{\frac{1}{\alpha}},u(+\infty)=0.$$

定理 2　若 $q<-(\alpha+1)$，$c=-\frac{\sqrt{\alpha+1}}{\alpha+1}(q+\alpha+1)$，$u^{\alpha}=\frac{-q}{1+e^{\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha q z}}$，方程(1)有行波解

$$u(z)=\left(\frac{-q}{1+e^{\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha q z}}\right)^{\frac{1}{\alpha}},\quad u(-\infty)=0.\ u(+\infty)=(-q)^{\frac{1}{\alpha}}.$$

定理 3　若 $q>-\frac{1}{\alpha+1}$，$c=\frac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1]$，$u^{\alpha}=\frac{1}{1+e^{-\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha z}}$. 则方程(1)有行波解

$$u(z)=\left(\frac{1}{1+e^{-\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha z}}\right)^{\frac{1}{\alpha}},\quad u(-\infty)=0.\ u(+\infty)=1.$$

定理 4　若 $q<-\frac{1}{\alpha+1}$，$c=-\frac{\sqrt{\alpha+1}}{\alpha+1}[(\alpha+1)q+1]$，$u^{\alpha}=\frac{1}{1+e^{\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha z}}$. 方程(1)有行波解

$$u(z)=\left(\frac{1}{1+e^{\frac{\sqrt{\alpha+1}}{\alpha+1}\alpha z}}\right)^{\frac{1}{\alpha}},\quad u(-\infty)=1.\ u(+\infty)=0.$$

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Travelling Wave Solutions of a Generalized Fisher Equation

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Abstract: In this paper, a generalized Fisher equation is discussed, and some explicit travelling wave solutions are obtained.

Key words: generalized Fisher equation; travelling wave solution; explicit travelling wave solution.