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Hamilton 算子和 Laplace 算子间的一个积分公式*

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摘 要：本文给出了 Hamilton 算子和 Laplace 算子间的一个积分公式，并指出文献[1]中的一个错误.

关键词：Hamilton 算子；Laplace 算子.
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定义 1 称 $\nabla \equiv i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}=\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\}$ 为 Hamilton 算子.

定义 2 称 $\Delta \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ 为 Laplace 算子.

不难发现 $\Delta=\nabla\cdot\nabla=\nabla^2$.

定理 设 $n(x,t)\in C^2(\mathbf{R}^3\times\mathbf{R},\mathbf{R})$，$\mathbf{r}=(r_1,r_2,r_3)\in\mathbf{R}^3$，$r=\sqrt{r_1^2+r_2^2+r_3^2}$，$V$ 是半径为 R 的球体，则有 $\int_V(\mathbf{r}\cdot\nabla)^2n(x,t)\mathrm{d}\mathbf{r}=\frac{1}{3}\nabla^2n(x,t)\int_V r^2\mathrm{d}\mathbf{r}=\frac{1}{3}\Delta n(x,t)\int_V r^2\mathrm{d}\mathbf{r}$.

证明

$$\int_V(\mathbf{r}\cdot\nabla)^2n(x,t)\mathrm{d}\mathbf{r}=\int_V\left[(r_1,r_2,r_3)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\right]^2n(x,t)\mathrm{d}\mathbf{r}$$

$$=\int_V\left[r_1^2\frac{\partial^2}{\partial x^2}+r_2^2\frac{\partial^2}{\partial y^2}+r_3^2\frac{\partial^2}{\partial z^2}+2r_1r_2\frac{\partial^2}{\partial x\partial y}+2r_1r_3\frac{\partial^2}{\partial x\partial z}+2r_2r_3\frac{\partial^2}{\partial y\partial z}\right]n(x,t)\mathrm{d}\mathbf{r}.$$

作球面坐标变换

$$\begin{cases}x=r\sin\varphi\cos\theta,\\ y=r\sin\varphi\sin\theta,\\ z=r\cos\varphi,\end{cases}$$

其中 $0\leqslant r\leqslant R$，$0\leqslant\varphi\leqslant\pi$，$0\leqslant\theta\leqslant2\pi$. 经计算得

$$\int_V r_1^2\frac{\partial^2n}{\partial x^2}\mathrm{d}\mathbf{r}=\frac{\partial^2n}{\partial x^2}\int_0^R\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_0^\pi r^2\sin^2\varphi\cos^2\theta\cdot r^2\sin\varphi\mathrm{d}\varphi=\frac{4}{15}\pi R^5\frac{\partial^2n}{\partial x^2},$$

$$\int_V r_2^2\frac{\partial^2n}{\partial y^2}\mathrm{d}\mathbf{r}=\frac{\partial^2n}{\partial y^2}\int_0^R\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_0^\pi r^2\sin^2\varphi\sin^2\theta\cdot r^2\sin\varphi\mathrm{d}\varphi=\frac{4}{15}\pi R^5\frac{\partial^2n}{\partial y^2},$$

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$$\int_V r_3^2 \frac{\partial^2 n}{\partial z^2}\mathrm{d}\mathbf{r} = \frac{\partial^2 n}{\partial z^2}\int_0^R \mathrm{d}r \int_0^{2\pi}\mathrm{d}\theta \int_0^\pi r^2\cos^2\varphi \cdot r^2\sin\varphi\mathrm{d}\varphi = \frac{4}{15}\pi R^5 \frac{\partial^2 n}{\partial z^2},$$

$$\int_V r_1 r_2 \frac{\partial^2 n}{\partial x\partial y}\mathrm{d}\mathbf{r} = \int_V r_1 r_3 \frac{\partial^2 n}{\partial x\partial z}\mathrm{d}\mathbf{r} = \int_V r_2 r_3 \frac{\partial^2 n}{\partial y\partial z}\mathrm{d}\mathbf{r} = 0.$$

于是

$$\begin{aligned}\int_V (\mathbf{r}\cdot\nabla)^2 n(x,t)\mathrm{d}\mathbf{r} &= \int_V \left[r_1^2\frac{\partial^2}{\partial x^2} + r_2^2\frac{\partial^2}{\partial y^2} + r_3^2\frac{\partial^2}{\partial z^2}\right] n(x,t)\mathrm{d}\mathbf{r}. \\ &= \frac{4}{15}\pi R^5\left[\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2}\right] = \frac{4}{15}\pi R^5 \nabla^2 n \\ &= \frac{4}{15}\pi R^5 \Delta n.\end{aligned}$$

又 $\int_V r^2\mathrm{d}\mathbf{r} = \int_0^R \mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_0^\pi r^2\cdot r^2\sin\varphi\mathrm{d}\varphi = \frac{4}{5}\pi R^5$，综上即得

$$\int_V (\mathbf{r}\cdot\nabla)^2 n(x,t)\mathrm{d}r = \frac{1}{3}\nabla^2 n(x,t)\int_V r^2\mathrm{d}\mathbf{r} = \frac{1}{3}\Delta n(x,t)\int_V r^2\mathrm{d}\mathbf{r}.$$

在 Murray J D 的 Mathematical Biology（见文献[1]）一书中，就出现了有关 Hamilton 算子和 Laplace 算子运算之间的一个错误. 该书 § 9. 5 Non-local Effects and Long Range Diffusion [p244－245]中等式(9. 34)的正确结果应如下：

$$\begin{aligned} n_{av} &= \left(\frac{3}{4\pi R^3}\right)\int_V\left[n(x,t) + \frac{1}{2}(\mathbf{r}\cdot\nabla)^2 n(x,t)\right]\mathrm{d}\mathbf{r} \\ &= \left(\frac{3}{4\pi R^3}\right)\left[n(x,t)\int_V \mathrm{d}\mathbf{r} + \frac{1}{6}\nabla^2 n(x,t)\int_V r^2\mathrm{d}\mathbf{r}\right] \\ &= n(x,t) + \frac{1}{10}R^2\nabla^2 n(x,t).\end{aligned}$$

于是，得到当 $R\to 0$ 时，$\nabla^2 n \propto \frac{\langle n(x,t)\rangle - n(x,t)}{R^2}$ 的正比例系数应为 10. 文献[1]中给出的正比例系数为 $\frac{10}{3}$，是一个严重错误.

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[1] MURRAY J D. *Mathematical Biology (2nd Edition)* [M]. Berlin: Springer-Verlay, 1989, 244－245.

An Integral Formula Involving Hamilton Operator and Laplace Operator

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Abstract: This paper gives an integral formula involving Hamilton operator and Laplace operator. Especially, we correct a mistake in [1].

Key words: Hamilton operator; Laplace operator.