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A Further Result of Nevanlinna's Four-Point Theorem *

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Abstract: In this paper, we proved a result that if two meromorphic functions share two values CM and two other values in the sense of $E_{k)}(\beta, f) = E_{k)}(\beta, g)$, $(k \geq 5)$, then f is a Möbius transformation of g.

Key words: meromorphic function; sharing value CM(IM).

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1. Introduction

In this paper the term "meromorphic function" will mean a meromorphic function in C. We will use the standard notations of Nevanlinna theory:

$$T(r,f), S(r,f), m(r,\beta,f), N(r,\beta,f), \overline{N}(r,\beta,f), N_1(r,\beta,f),$$

$$\overline{N}_1(r,\beta,f), N_1(r,f), \overline{N}(r,f), \Theta(\beta,f), (\beta \in C \cup \{\infty\}), \cdots,$$

and we assume that the reader is familiar with the basic results in Nevanlinna theory as found in [3].

For a nonconstant meromorphic function f, a number $\beta \in C \cup \{\infty\}$ and a positive integer k (or $+\infty$), we write $E_{k)}(\beta, f)$ for the set of zeros of $f(z) - a$ with multiplicity $\leq k$ (counting multiplicity); we write $\overline{E}_{k)}(\beta, f)$ for the set of zeros of $f(z) - a$ with multiplicity $\leq k$ (each zero counted only once).

If two nonconstant meromorphic function f and g satisfy $E_{+\infty)}(\beta, f) = E_{+\infty)}(\beta, g)$, then we say that f and g share βCM; If f and g satisfy $\overline{E}_{+\infty)}(\beta, f) = \overline{E}_{+\infty)}(\beta, g)$, then we say that f and g share βIM.

Gundersen[2] proved the following result which generalizes a well-known result of Nevanlinna called *Four-Point Theorem*[5].

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Theorem A *Let f and g be nonconstant meromerphic functions. Assume that f and g share two values 0 and ∞ CM, and that they share two values 1 and $a(\neq 0, \infty, 1)$ IM.*

(i) If $a = -1$, then $fg \equiv 1$, or $f + g \equiv 0$, or $f \equiv g$.

(ii) If $a = 1/2$, then $(f - (1/2))(g - (1/2)) \equiv 1/4$, or $f + g \equiv 1$ or $f \equiv g$.

(iii) If $a = 2$, then $(f - 1)(g - 1) \equiv 1$, or $f + g \equiv 2$ or $f \equiv g$.

(iv) If $a \neq -1, 1/2, 2$, then $f \equiv g$.

Because as we know if f and g share two values 0 and ∞ CM, and share two other values 1 and $a(\neq 0, \infty, 1)$ IM, they share all four values CM (See [5]). Theorem A is equivalent to say f and g share all four values CM and we get the same result.

In 1998, Hideharu Ueda got the following result.

Theorem B[1] *Let f and g be two non-constant meromorphic functions. Assume that f and g share two values 0 and ∞ CM, and that they satisfy $E_{k)}(a_j, f) = E_{k)}(a_j, g)$ for $j = 3, 4$, where $a_3 = 1, a_4 = a(\neq 0, \infty, 1,)$ and $k(\geq 12)$ is a positive integer. Then f and g satisfy one of the following cases:*

(i) $f \equiv g$;

(ii) $f \equiv -g$ and $a = -1$;

(iii) $f \equiv -g + 2$ and $a = 2$;

(iv) $(f - \frac{1}{2})(g - \frac{1}{2}) = \frac{1}{4}$ and $a = \frac{1}{2}$;

(v) $fg \equiv 1$ and $a = -1$;

(vi) $(f - 1)(g - 1) \equiv 1$ and $a = 2$;

(vii) $f \equiv -g + 1$ and $a = \frac{1}{2}$.

In [1], H. Ueda points out that he doesn't know if we can still keep the result in Theorem B when $2 < k < 12$. In this paper, we obtain the following result, which tells that we can keep the result in Theorem B when $k(\geq 5)$. So it is an improvement of Theorem B .

Theorem 1 *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values 0 and ∞ CM, and that they satisfy $E_{k)}(a_j, f) = E_{k)}(a_j, g)$ for $j = 3, 4$, where $a_3 = 1, a_4 = a(\neq 0, \infty, 1, -1,)$ and $k(\geq 5)$ is a positive integer.*

(i) If $a = -1$, then $fg \equiv 1$, or $f + g \equiv 0$, or $f \equiv g$.

(ii) If $a = 1/2$, then $(f - (1/2))(g - (1/2)) \equiv 1/4$ or $f + g \equiv 1$ or $f \equiv g$.

(iii) If $a = 2$, then $(f - 1)(g - 1) \equiv 1$ or $f + g \equiv 2$ or $f \equiv g$.

(iv) If $a \neq -1, 1/2, 2$, then $f \equiv g$.

2. Notations and terminologies

In this Section, we introduce some essential notations and terminologies.

(i) Let f, g be distinct nonconstant meromorphic functions. For $r > 0$, let $T(r) = \max\{T(r,f), T(r,g)\}$. We write $\sigma(r) = S(r)$ for every function $\sigma : (0, \infty) \to (-\infty, +\infty)$ satisfying $\sigma(r)/T(r) \to 0$ for $r \to +\infty$ possibly outside a set of finite Lebesgue measure.

(ii) Let f, g be nonconstant meromorphic functions. we denote by $\overline{N}_c(r, \beta; f, g) \equiv \overline{N}_c(r, \beta)$ (resp. $\overline{N}_d(r, \beta; f, g) \equiv \overline{N}_d(r, \beta)$) the counting function of those common β-points of f and g with the same multiplicity (resp. with the different multiplicities), each point counted only once regardless of multiplicity, and we write

$$\overline{N}_i(r,\beta;f,g)\equiv\overline{N}_i(r,\beta)=\overline{N}_c(r,\beta)+\overline{N}_d(r,\beta).$$

We say that f and g share $\beta\, CM''$ if $\overline{N}(r,\beta,f)-\overline{N}_c(r,\beta)=S(r,f)$ and $\overline{N}(r,\beta,g)-\overline{N}_c(r,\beta)=S(r,g)$ hold. Similarly, if $\overline{N}(r,\beta,f)-\overline{N}_i(r,\beta)=S(r,f)$ and $\overline{N}(r,\beta,g)-\overline{N}_i(r,\beta)=S(r,g)$ hold, then we say that f and g share $\beta\, IM''$. These notions CM'' and IM'' are slight generalizations of CM and IM , respectively.

(iii) Let f and g be nonconstant meromorphic functions. For $\beta,\gamma(\in C\cup\{\infty\}),\beta\neq\gamma$ we put

$$\begin{aligned}
m_{\beta,\gamma}(r)\equiv m_{\beta,\gamma}(r;f,g)&=m(r,\beta,f)+m(r,\gamma,f)+m(r,\beta,g)+m(r,\gamma,g),\\
\overline{N}_{\beta,\gamma}(r)\equiv\overline{N}_{\beta,\gamma}(r;f,g)&=\overline{N}(r;f=\beta,g\neq\beta)+\overline{N}(r;f=\gamma,g\neq\gamma)+\\
&\quad\overline{N}(r;g=\beta,f\neq\beta)+\overline{N}(r;g=\gamma,f\neq\gamma),\\
\tilde{N}'_{\beta,\gamma}(r)\equiv\tilde{N}'_{\beta,\gamma}(r;f,g)&=\overline{N}_c(r,\beta)+\overline{N}_c(r,\gamma),\\
\tilde{N}''_{\beta,\gamma}(r)\equiv\tilde{N}''_{\beta,\gamma}(r;f,g)&=\overline{N}_d(r,\beta)+\overline{N}_d(r,\gamma),
\end{aligned}$$

$$\tilde{N}_{\beta,\gamma}(r)\equiv\tilde{N}_{\beta,\gamma}(r;f,g)=\tilde{N}'_{\beta,\gamma}(r;f,g)+\tilde{N}''_{\beta,\gamma}(r;f,g)=\overline{N}_i(r,\beta;f,g)+\overline{N}_i(r,\gamma;f,g),$$

where for example, $\overline{N}(r;f=\beta,g\neq\beta)$ denotes the counting function of those $\beta-$ points of f which are not $\beta-$points of g ,each point counted only once.

3. Preparations for the proof of Theorems 1

We need a slight generalization of Theorem A:

Theorem A' *Theorem A remains still valid if CM and IM are replaced by CM″ and IM″, respectively.*

In order to prove this fact we only need to use the argument (due to Mues) in the proof of Theorem 1 in [4] by replacing CM and IM by CM″ and IM″, respectively.

In the rest of this section, we assume that f and g are distinct nonconstant meromorphic functions sharing $a_1=0$ and $a_2=\infty$ CM and satisfying $E_{k)}(a_j,f)=E_{k)}(a_j,g)$ for $j=3,4,$ where $a_3=1,a_4=a(\neq 0,\infty,1)$ and $k(\geq 2)$ is a positive integer. We write, for example,

$$N(r,0,f)=N(r,0,g)=N(r,0),$$

$$N(r,\infty,f)=N(r,\infty,g)=N(r,\infty),$$

$$\overline{N}(r,0,f)=\overline{N}(r,0,g)=\overline{N}(r,0),$$

$$\overline{N}(r,\infty,f)=\overline{N}(r,\infty,g)=\overline{N}(r,\infty),$$

$$N_1(r,0,f)=N_1(r,0,g)=N_1(r,0),$$

$$N_1(r,\infty,f)=N_1(r,\infty,g)=N_1(r,\infty),$$

$$\overline{N}_1(r,0,f)=\overline{N}_1(r,0,g)=\overline{N}_1(r,0),$$

$$\overline{N}_1(r,\infty,f)=\overline{N}_1(r,\infty,g)=\overline{N}_1(r,\infty).$$

Lemma 1[6] $S(r) = S(r,f) = S(r,g)$.

Lemma 2[1] *Let* $\tilde{n}(r; f' = g' = 0, f \neq 0, g \neq 0)$ *denote the number of distinct common zeros of* f' *and* g' *which are neither zeros of* f *nor* g *in* $|z| \leq r$. *Put* $\tilde{N}(r; f' = g' = 0, f \neq 0, g \neq 0) = \int_0^r \{\tilde{n}(t; f' = g' = 0, f \neq 0, g \neq 0) - \tilde{n}(0; f' = g' = 0, f \neq 0, g \neq 0)\}/t dt + \tilde{n}(0; f' = g' = 0, f \neq 0, g \neq 0) \log r$. *If* g/f *is not a constant, then*

$$\tilde{N}(r; f' = g' = 0, f \neq 0, g \neq 0) = S(r).$$

Lemma 3[6] *Let* $n_1'(r,f)$ *denote the number of multiple points of* f *in* $|z| \leq r$ *such that* $f \neq 0, \infty, 1, a$, *where a point of multiplicity* m *is counted* $(m-1)$ *times. Put* $N_1'(r,f) = \int_0^r \{n_1'(t,f) - n_1'(0,f)\}/t dt + n_1'(0,f) \log r$. *If* $N_1'(r,g)$ *is similarly defined, then*

$$\begin{aligned} &\tilde{N}_{1,a}''(r; f, g) + k\overline{N}_{1,a}(r; f, g) + N_1'(r,f) + N_1'(r,g) \\ &\leq 2\{\overline{N}(r,0) + \overline{N}(r,\infty)\} + S(r). \end{aligned} \tag{3.1}$$

Now, we introduce some auxiliary functions:

$$\varphi_1 = \frac{f'f}{(f-1)(f-a)} - \frac{g'g}{(g-1)(g-a)}, \tag{3.2}$$

$$\varphi_2 = \frac{f'}{f(f-1)(f-a)} - \frac{g'}{g(g-1)(g-a)}. \tag{3.3}$$

With the aid of these auxiliary functions, we obtain some basic conclusions:

Lemma 4[6]

(i)

$$2\{N_1(r,0) + N_1(r,\infty)\} + \overline{N}_1'(r,f) + \overline{N}_1'(r,g) \leq \overline{N}_{1,a}(r) + S(r). \tag{3.4}$$

(ii) If neither $\varphi_1 \equiv 0$ *nor* $\varphi_2 \equiv 0$, *then*

$$\overline{N}(r,0) + \overline{N}(r,\infty) \leq 2\{\overline{N}_{1,a}(r) + \tilde{N}_{1,a}''(r)\} + S(r). \tag{3.5}$$

4. Proof of Theorem 1

In what follows we assume that f and g are distinct and satisfy the assumptions of Theorem 1 , and so there is an entire function α satisfying $g = e^{\alpha} f$ $(e^{\alpha} \not\equiv 1)$.

Case 1 We first consider the case that e^{α} is a constant $C\,(\neq 0, 1)$. From the assumptions $E_{k)}(1,f) = E_{k)}(1,g)$ and $E_{k)}(a,f) = E_{k)}(a,g)$ it follows that

$$\Theta(1,g), \Theta(a,g) \geq k/(k+1).$$

If $C \neq a$, we also obtain

$$\Theta(C,g) \geq k/(k+1),$$

and so

$$\Theta(1,g)+\Theta(a,g)+\Theta(C,g) \geq 3k/(k+1) > 2,$$

a contradiction. This shows $C = a$. Further if $a^2 \neq 1$, we also obtain

$$\Theta(a^2,g) \geq k/(k+1),$$

and so

$$\Theta(1,g)+\Theta(a,g)+\Theta(a^2,g) \geq 3k/(k+1) > 2,$$

a contradiction. This shows $a^2 = 1$, i.e.,$a = -1$ and $f + g \equiv 0$. In this case we remark that

$$N(r,1,f) = N(r,-1,g), N(r,-1,f) = N(r,1,g)$$

are not necessarily $S(r)$!

Case 2 We next consider the case that e^{α} is nonconstant. We divide our argument into several subcases:

2.1. The case $\varphi_1 \equiv 0$

$\varphi_1 \equiv 0$ implies that any 1-and $a-$ point of f (resp. g) are a 1- or an $a-$ point of g (resp. f). By Lemma 2, we deduce from the assumptions $E_{k)}(a_j,f) = E_{k)}(a_j,g)$ for $j = 3,4$ with $a_3 = 1, a_4 = a$ that $\overline{N}(r; f = 1, g = a) + \overline{N}(r; f = a, g = 1) = S(r)$, (where $\overline{N}(r; f = 1, g = a)$ denote the counting function of common roots of $f = 1$ and $g = a$, each counted only once,) and so by Lemma 1, f and g share two values 1 and a IM''. Hence by Theorem A', f and g are connected with one of the relations stated in Theorem A. Further, straightforward computations show that only two relations $(f-(1/2))(g-(1/2)) \equiv 1/4$ (with $a = 1/2$) and $(f-1)(g-1) \equiv 1$ (with $a = 2$) are suitable for $\varphi_1 \equiv 0$.

2.2. The case $\varphi_2 \equiv 0$

The same reasoning as in the case 2.1 shows that only two relations $f+g \equiv 2$, (with $a = 2$) and $f + g \equiv 1$ (with $a = 1/2$) are suitable for $\varphi_2 \equiv 0$.

2.3. The case $\varphi_1 \not\equiv 0, \varphi_2 \not\equiv 0$

From Lemma 2, noting that $E_{k)}(a_j,f) = E_{k)}(a_j,g)$, $(j = 3,4)$ we can get

$$\tilde{N}''_{1,a}(r) = S(r).$$

In fact, since $\tilde{N}''_{1,a}(r) \equiv \tilde{N}''_{1,a}(r; f,g) = \overline{N}_d(r,1) + \overline{N}_d(r,a)$, and noting that $E_{k)}(a_j,f) = E_{k)}(a_j,g)$, $(j = 3,4)$ we have, for any common 1- (resp. a-)points z_0 of f and g with the different multiplicities, both of the multiplicities are more than $k(\geq 5)$. So we have $f'(z_0) = g'(z_0) = 0$, $f(z_0) = 1$ (resp. a) $\neq 0$ and $g(z_0) = 1$ (resp. a) $\neq 0$. Hence from Lemma 2, if g/f is not a constant, we can get

$$\tilde{N}''_{1,a}(r) \equiv \tilde{N}''_{1,a}(r; f,g) = \overline{N}_d(r,1) + \overline{N}_d(r,a) \leq \tilde{N}(r; f' = g' = 0, f \neq 0, g \neq 0) = S(r).$$

If g/f is a constant, since f and g share two values 0 and ∞ CM, we can easily get that $f \equiv g$.

Combining (3.1) and (3.5), we have

$$(k-4)\overline{N}_{1,a}(r) \leq 3\tilde{N}''_{1,a}(r) + S(r). \tag{4.1}$$

Taking the fact $k \geq 5$ into account, we deduce that $\overline{N}_{1,a}(r) = S(r)$.

Hence, $\tilde{N}''_{1,a}(r) = S(r)$ and $\overline{N}_{1,a}(r) = S(r)$ hold . From (3.4) and (3.5) we obtain $N(r,0) + N(r,\infty) = S(r)$, and so by Lemma 1 and the second fundamental theorem $\overline{N}(r,1,f), \overline{N}(r,a,f) = T(r,f) + S(r)$ and $\overline{N}(r,1,g), \overline{N}(r,a,g) = T(r,g) + S(r)$. On the other hand, $\overline{N}_{1,a}(r) = S(r)$ implies that f and g share two values 1 and a IM'', and so we deduce from Theorem A' that f and g are connected with one of the relations in Theorem A. Therefore we obtain $fg \equiv 1$ with $a \equiv -1$ in this case.

This completes the proof of Theorem 1 .

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关于 Nevanlinna 四值定理的改进

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摘 要: 在本文中，我们证明了如果两个亚纯函数分担两个值 CM, 并且在 $E_{k)}(\beta, f) = E_{k)}(\beta, g) (k \geq 5)$, 意义下分担另外两个值，则这两个亚纯函数一个是另一个的分式线性变换.

关键词: 亚纯函数；小函数；唯一性.