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On Sectional Cycles in Translation Quivers *

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Abstract: In this paper, we prove that there exists no sectional cycle in a translation quiver under certain conditions. So, we generalize Bautista and Smalϕ's corresponding result on AR-quiver of an artin algebra.

Key words: sectional cycles; translation quivers; additive length functions.

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In [1], Bautista and Smalϕ proved a well-known result.

Bautista and Smalϕ's Theorem *Let $\wedge$ be any artin algebra . Then the Auslander-Reiten quiver of $\wedge$ never contains a sectional path which is a cycle.*

In this paper, we will show that Bautista and Smalϕ's proof can be translated into a purely combinatorial one. Before we state our combinatorial result for the above theorem, let us fix some terminology.

Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver without loops and multiple arrows, where Γ_0 is the set of vertices, Γ_1 is the set of arrows and $\tau : \Gamma_0' \to \Gamma_0$ is an injective map for some subset $\Gamma_0' \subseteq \Gamma_0$. Given a vertex x, denote by x^+ the set of vertices y such that there is an arrow $x \to y$; the set x^- consists of all vertices y such that there is an arrow $y \to x$. A vertex x with $\tau^t x = x$ for some positive integer $t \geq 1$ is said to be periodic. Let $\delta : \Gamma_1 \longrightarrow \mathbf{N} \times \mathbf{N}$ be a map and denote the values by $\delta(\alpha) = (\delta_{x,y}, \delta'_{x,y})$ for each arrow $\alpha : X \longrightarrow Y$, where $\mathbf{N}$ is the set of natural numbers. The triple (Γ, τ, δ) is called a valued translation quiver if the following conditions are satisfied for all non-projective vertices x : (1). $\delta'_{\tau x,y} = \delta_{y,x}$ for all $y \in x^-$. (2). $\delta_{\tau x,y} = \delta'_{y,x}$ for all $y \in x^-$. For any x , y in Γ , if $\delta_{x,y} = \delta'_{x,y} = 1$, Γ is said to be trivially valued. A map $l : \Gamma_0 \to \mathbf{N}$ (the set of natural numbers) is called an additive length function for $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ if the following conditions are satisfied for all vertices x,

1). $l(x) + l(\tau x) = \sum_{y \in x^-} \delta_{y,x} l(y)$, if x is non-projective;

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2). $l(x) > \sum_{y\in x^-} \delta_{y,x} l(y)$, if x is projective;
3). $l(x) > \sum_{y\in x^+} \delta'_{y,x} l(y)$, if x is injective.

Given a quiver Δ , we now define its path category as follows: it is an additive category, with objects being direct sums of indecomposable objects. The indecomposable objects in the path category of Δ are given by the vectices of Δ, and given a , b $\in \Delta$,the set of maps from a to b is given by the $k-$vector space with basis the set of all paths from a to b . The composition of maps is induced from the usual composition of paths: $(a|\alpha_1 ,\cdots , \alpha_l |b)(b|\beta_1 ,\cdots ,\beta_s |c) = (a|\alpha_1 ,\cdots , \alpha_l ,\beta_1 ,\cdots ,\beta_s |c)$, where $(a|\alpha_1 ,\cdots , \alpha_l |b)$ is a path from a to b, and $(b|\beta_1 ,\cdots ,\beta_s |c)$ is a path from b to c.

Given a translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau)$, a polarization of Γ is given by an injective map $\sigma : \Gamma_1' \longrightarrow \Gamma_1$ where Γ_1' is the set of all arrows $\alpha : a \longrightarrow b$ with b not projective, such that $\sigma(\alpha) : \tau b \longrightarrow a$ for $\alpha : a \longrightarrow b$. In case Γ has no multiple edges, there is a unique polarization. Given a translation quiver Γ , its mesh category $k(\Gamma, \sigma)$ can be defined as follows. First, we define the mesh ideal in the path category of (Γ_0, Γ_1) as the ideal generated by the elements

$$m_z = \sum_{y\in z^-} \sum_{\sigma: y\rightarrow z} \sigma(\alpha)\alpha$$

with z a non-projective vertex. The mesh category $k(\Gamma, \sigma)$ is defined as the quotient category of the path category of (Γ_0, Γ_1) modulo the mesh ideal. In case Γ is a translation quiver without multiple edges, σ is uniquely determined by Γ , thus we denote the corresponding mesh category just by $k(\Gamma)$.

We define the radical of the mesh category $k(\Gamma)$ as follows: If X , Y are indecomposable (that is , they are vertices in Γ), let $\mathrm{rad}(X, Y)$ be the set of non-invertible morphism from X to Y. The powers of the radical are defined inductively as $\mathrm{rad}^{i+1}(X, Y) = \{f \in \mathrm{Hom}_{k(\Gamma)}(X, Y) \mid \exists M \in k(\Gamma)$ and $g \in \mathrm{rad}^i(X, M), h \in \mathrm{rad}(M, Y)$ with $f = hg\}$. Now , the infinite radical is defined as $\mathrm{rad}^\infty(X, Y) = \bigcap\limits_{i<\infty} \mathrm{rad}^i(X, Y)$.

A sectional path is a chain of vertices x_i and arrows $x_i \longrightarrow x_{i+1}$ in Γ such that $x_{i+2} \neq \tau^{-1} x_i$. A sectional path which is an oriented cycle will be called a sectional cycle.

Theorem *Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a connected translation quiver without loops and multiple arrows. Assume there is an additive length function on a translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ with the property that $l(x) \neq l(y)$, for any arrow $X \longrightarrow Y$ in Γ, and $k(\Gamma)$ is the mesh-category. Assume moreover that for any $x \in \Gamma_0$ there exists a natural number $n(x)$ such that $\mathrm{rad}^{n(x)}(x, x) = 0$. Then there is no sectional cycle in Γ.*

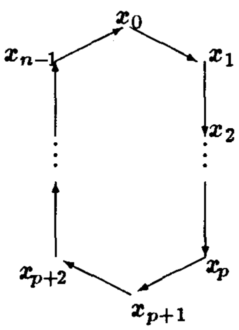
For any artin algebra Λ, let modΛ denote the category of finitely generated $\Lambda-$modules. Then we know that the radical of $\mathrm{End}(A)$ is nilpotent for any $\Lambda-$module A in modΛ[2]. So, our theorem implies Bautista and smalϕ's theorem since there is an additive length function on $AR-$quiver $\Gamma(\Lambda)$ of the artin algebra Λ , and for any irreducible morphism $f : A \longrightarrow B$ in $\Gamma(\Lambda)$, f is either monomorphism or epimorphism and thus $l(A) \neq l(B)$.

Now we are going to prove our theorem. In the following we assume that Γ is the translation quiver which satisfies the conditions in our theorem. It suffices to prove the non-existence of a minimal sectional cycle in the translation quiver which satisfies the conditions in our theorem. Hence, when we refer to a sectional cycle, we will always assume the following indexing

$$x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longrightarrow x_0,$$

where $x_0 = x_n$ and $x_i \neq x_j$ when $i \neq j$ and $i\ ,j < n$. Thus we only need to give the indices modulo n when we are dealing with a sectional cycle. We can use the following picture.

$\mathcal{X}$:



Lemma 1 *Let $\mathcal{X}$ be a sectional cycle in Γ, then no x_i is injective or projective.*

Proof Let us choose i such that $l(x_i)$ is minimal in the set $\{l(x_j) \mid x_j \text{ in } \mathcal{X}, \text{ for } j = 0,1,2,\cdots,n-1\}$. As the arrow $x_i \longrightarrow x_{i+1}$, we know that $l(x_i) < l(x_{i+1})$. Then x_i is not injective, furthermore, as $l(\tau^- x_i) + l(x_i) \geq \delta_{x_{i+1},x_i} l(x_{i+1})$, we have that $l(\tau^- x_i) \geq \delta_{x_{i+1},x_i} l(x_{i+1}) - l(x_i) \geq l(x_{i+1}) - l(x_i)$. Hence , $l(x_{i+2}) + l(\tau^- x_i) - l(x_{i+1}) \geq l(x_{i+2}) + l(x_{i+1}) - l(x_i) - l(x_{i+1} = l(x_{i+2})) - l(x_i) \geq 0$.

Thus, x_{i+1} is not injective and $l(\tau^- x_{i+1}) \geq l(x_{i+2}) + l(\tau^- x_i) - l(x_{i+1}) \geq l(x_{i+2}) - l(x_i)$. Now by induction on j we get $l(x_{i+j+2}) + l(\tau^- x_{i+j}) - l(x_{i+j+1}) \geq l(x_{i+j+2}) + l(x_{i+j+1}) - l(x_i) - l(x_{i+j+1}) = l(x_{i+j+2}) - l(x_i) \geq 0$. So, x_{i+j+2} is not injective for any j.

Dually, we can show that no x_i is projective.

Lemma 2 *If there exists a sectional cycle $\mathcal{X}$ in Γ, then the meshes in Γ are of the form*

$$\begin{array}{ccccc} & & \tau^{-(k+1)} x_{i-1} & & \\ & \nearrow & & \searrow & \\ \tau^{-k} x_i & & & & \tau^{-(k+1)} x_i\ , \quad k \in \mathbf{Z}. \\ & \searrow & & \nearrow & \\ & & \tau^{-k} x_{i+1} & & \end{array}$$

Thus, Γ contains neither projective vertices nor injective vertices, and all vertices in Γ are of the form $\tau^{-k} x_i$, $i = 0, 1, \cdots, n-1$, and $k \in \mathbf{Z}$, and Γ is trivially valued.

Proof Consider the sectional cycle $\mathcal{X}$, then by Lemma 1, no x_i is projective or injective and hence, τ^-– and τ–translates of the sectional cycle is a sectional cycle. So, by induction on k using Lemma 1, we get that the τ^{-k}– and τ^k–translates of the sectional cycle are sectional cycles. Thus, for any $k \in \mathbf{Z}$ and $i = 0, 1, \cdots, n-1$, $\tau^{-k} x_i$ is neither projective nor injective. Further, for any $k \in \mathbf{Z}$ and $i = 0, 1, \cdots, n-1$, we have the follwing sub-quiver,

$$\begin{tikzcd}[column sep=small] & \tau^{-k}x_{i+1} \arrow[dr] & \\ \tau^{-k}x_i \arrow[ur] \arrow[dr] & & \tau^{-(k+1)}x_i. \\ & \tau^{-(k+1)}x_{i-1} \arrow[ur] & \end{tikzcd}$$

Thus, we have

$$l(\tau^{-k}x_i)+l(\tau^{-(k+1)}x_i)) \geq \delta_{\tau^{-k}x_{i+1},\tau^{-(k+1)}x_i} l(\tau^{-k}x_{i+1}) + \delta_{\tau^{-(k+1)}x_{i-1},\tau^{-(k+1)}x_i} l(\tau^{-(k+1)}x_{i-1})$$

for all $k \in \mathbf{Z}$ and $i = 0, 1, \cdots, n-1$, where $\delta_{\tau^{-k}x_{i+1},\tau^{-(k+1)}x_i} \geq 1$ and $\delta_{\tau^{-(k+1)}x_{i-1},\tau^{-(k+1)}x_i} \geq 1$.

Now, by keeping k fixed and summing over all indices $i \in \{0, 1, \cdots, n-1\}$, we get that this has to be an equality, showing that the subquivers are complete meshes and $\delta_{\tau^{-k}x_{i+1},\tau^{-(k+1)}x_i} = 1 = \delta_{\tau^{-(k+1)}x_{i-1},\tau^{-(k+1)}x_i}$.

Therefore, we get that the set $\{\tau^{-k}x_i \mid i = 0, 1, \cdots, n-1, k \in \mathbf{Z}\}$ is the whole set Γ_0 of the translation quiver Γ since the set does not contain injectives or projectives, and is closed with respect to meshes and Γ is trivially valued.

Lemma 3 *If there is a sectional cycle $\mathcal{X}$ in Γ, then no x_i is τ-periodic.*

Proof By Lemma 2, the translation quiver Γ is stable, and every mesh is a rectangle. So, if one of x_i is τ-periodic, by using Happel-Preiser-Ringel theorem[5] and Lemma 2, we know that $\Gamma \simeq \mathbf{Z}\widetilde{\mathbf{A}}_{\mathbf{t}}/(\tau^{\mathbf{m}})$. Hence , we get that $\mathrm{rad}^m(x_0, x_0) \neq 0$ for any natural number m since $x_0 \to x_1 \to x_2 \to \cdots \to x_{n-1} \to x_0$ is sectional. This contradicts the condition that there is a natural number $n(x)$ such that $\mathrm{rad}^{n(x)}(x, x) = 0$ for any x. So, no x_i is periodic in $\mathcal{X}$.

Lemma 4 *Let $\mathcal{X}$ be a sectional cycle in Γ. Then*

a). for any $p,\ q \in \{0, 1, 2, \cdots, n-1\}$ and $0 \neq k \in \mathbf{Z}$, we have that $x_p \not\simeq \tau^{-k}x_q$.

b). $\mathrm{Hom}_{k(\Gamma)}(x_p, \tau x_q) = \mathrm{rad}^{\infty}_{k(\Gamma)}(x_p, \tau x_q)$ for all $p, q \in \{0, 1, 2, \cdots, n-1\}$.

Proof a). Assume that $x_p \simeq \tau^{-k}x_q$ for $0 \neq k \in \mathbf{Z}$ and $p, q \in \{0, 1, 2 \cdots, n-1\}$. Then $p \neq q$ by Lemma 3. Further, if $x_p \simeq \tau^{-k_1}x_q$ and $x_p \simeq \tau^{-k_2}x_q$ with $k_1, k_2 \in \mathbf{Z}$, $k_1 \neq 0, k_2 \neq 0$, then $x_q \simeq \tau^{k_1-k_2}x_q$. Hence $k_1 = k_2$ by Lemma 3. Therefore, for each pair p, q in $\{0, 1, 2 \cdots, n-1\}$, there exists at most one k such that $x_p \simeq \tau^{-k}x_q$. So, we have the following mesh in Γ ,

$$\begin{tikzcd}[column sep=small] & x_{p+1} \arrow[dr] & \\ \tau^{-k}x_q \simeq x_p \arrow[ur] \arrow[dr] & & \tau^{-(k+1)}x_q. \\ & \tau^{-}x_{p-1} = \tau^{-(k+1)}x_{q-1} \arrow[ur] & \end{tikzcd}$$

Hence by Lemma 2, x_{p+1} is either isomorphic to $\tau^{-k}x_{q+1}$ or isomorphic to $\tau^{-(k+1)}x_{q-1}$. By the choice of k, we have $x_{p+1} \simeq \tau^{-k}x_{q+1}$. By induction on j we get that $x_{p+j} \simeq \tau^{-k}x_{q+j}$

and therefore τ^{-k} permutes the set $\{x_0, x_1, \cdots, x_{n-1}\}$. Then some power of τ^{-k} fixes the set, this contradicts Lemma 3.

b). By a) we get that there is a chain of arrows from $\tau^{-k_1}x_j$ to $\tau^{-k_2}x_i$ if and only if $k_2 \geq k_1$. Thus , there is no chain of arrows from x_i to τx_j for any i and j in $\{0, 1, \cdots, n-1\}$. Hence , b) follows.

Now , we are ready to prove our theorem.

Assume that Γ contains a sectional cycle $\mathcal{X}$. Let $\mathcal{X}$ contain n vertices, then we have $\mathcal{X}$ in the form $x_0 \to x_1 \to \cdots \to x_{n-1} \to x_0$. From Lemma 1-4, we know that $\mathcal{C} \simeq \mathbf{Z}\widetilde{A}_{n-1}$. Thus, we know $\mathrm{rad}^m(x_0, x_0) \neq 0$ for any natural number m since $x_0 \to x_1 \to \cdots \to x_{n-1} \to x_0$ is sectional . This contradicts the condition that for any x there exists a natural number $n(x)$ such that $\mathrm{rad}^{n(x)}(x, x) = 0$. This completes the proof.

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关于平移箭图上的截点圈

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摘 要: 在本文中，我们证明了在一定条件下平移箭图中不存在截点圈 (sectional cycle), 从而推广了在阿丁代数的 AR- 箭图上 Bautista 和 Smalϕ 的相应结果.

关键词: 截点圈；平移箭图；加性长度函数.