Journal of Mathematical Research & Exposition
Vol.23, No.3, 443–448, August, 2003

Admissible Meromorphic Solutions of a Type of Higher-Order Algebraic Differential Equation *

GAO Ling-yun
(Dept. of Math., Jinan University, Guangzhou 510632, China)

Abstract: Using Nevanlinna theory of the value distribution of meromorphic functions, we investigate the form of a type of algebraic differential equation with admissible meromorphic solutions and obtain a Malmquist type theorem.

Key words: meromorphic admissible solutions; algebraic differential equations; finite accumulations.

Classification: AMS(2000) 34A34,30D05,30D35/CLC number: O174.52

Document code: A **Article ID:** 1000-341X(2003)03-0443-06

1. Introduction and main result

In 1978, N.Steinmetz investigated the existence problem of admissible solutions of algebraic differential equation of the form

$$\Omega(z,w) = H(z,w), \tag{1}$$

where $\Omega(z,w) = \sum_{(i)} a_{(i)}(z) w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$, $H(z,w)$ is quotient of entire function in variables z and w. He obtained

Theorem A[2] *If the differential equation (1) admits an admissible meromorphic solution $w(z)$, then (1) must be degenerate into a polynomial in w and*

$$\deg_w^{H(z,w)} \le \Delta,$$

where $\Delta = \max\{i_0 + 2i_1 + \ldots + (n+1)i_n\}$.

In this paper we will consider the existence of admissible solution of gerenal algebraic differential equations of the form

$$(\Omega_1(z,w))/(\Omega_2(z,w)) = H(z,w), \tag{2}$$

*Received date: 2000-12-20
Foundation item: Supported by NNSF of China (19971052)
Biography: GAO Ling-yun (1963-), male, Ph.D., Associate Professor.

where $\Omega_1(z,w)=\sum_{(i)} a_{(i)}(z)w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$, $\Omega_2(z,w)=\sum_{(j)} b_{(j)}(z)w^{j_0}(w')^{j_1}\cdots(w^{(n)})^{j_n}$ are differential polynomials with meromorphic coefficients $\{a_{(i)}\}$ and $\{b_{(j)}\}$ respectively, $(i),(j)$ are two finite index sets, $H(z,w)$ is meromorphic function in z and w.

For differential polynomial $\Omega_1(z,w),\Omega_2(z,w)$, we adopt the following notations.

$$\lambda_1=\max\{\sum_{l=0}^{n} i_l\},u_1=\max\{\sum_{l=1}^{n} l i_l\},\Delta_1=\max\{\sum_{l=0}^{n}(l+1)i_l\},$$

$$\lambda_2=\max\{\sum_{l=0}^{n} j_l\},u_2=\max\{\sum_{l=1}^{n} l j_l\},\Delta_2=\max\{\sum_{l=0}^{n}(l+1)j_l\}.$$

Definition *put* $S_1(r)=\sum_{(i)}T(r,a_{(i)})+\sum_{(j)}T(r,b_{(j)}),S_c(r)=T(r,H(z,c)),c\in\mathbf{C}.\ E=\{c\}$ *is a finite accumulation set in the complex plane. Let* $w(z)$ *be a meromorphic solution of (2) and* I *be a set of* r *of finite linear measure. For every such* $c\in E$, *if* $\limsup_{r\to\infty,r\notin I}(S_1(r)+S_c(r))/T(r,w)=0$, *we say that* $w(z)$ *is an admissible solution of (2).*

The following result,as well as its proof, need some familiarity with the Nevanlinna theory, see, e.g.[1] for notations and basic results.

Our main result is:

Theorem 1 *If* $w(z)$ *is an admissible meromorphic solutions of (2), then* $H(z,w)$ *must be rational function in* w,*and the degree of* w *satisfies*

$$\deg_w^{H(z,w)}\le\lambda+(\Delta-\lambda)(1-\theta(w,\infty))\le\Delta,$$

where $\lambda=\max\{\lambda_1,\lambda_2\},\Delta=\max\{\Delta_1,\Delta_2\},\theta(w,\infty)=1-\limsup\frac{\overline{N}(r,w)}{T(r,w)}$.

2. Proof of Theorem 1

Let $w(z)$ be an admissible meromorphic solutions of (2). For $c_1\in E$. Set

$$\varphi_1(z;c_1)=\frac{\Omega_1}{H(z,c_1)(w-c_1)}-\frac{\Omega_2}{w-c_1}=\frac{\Omega_1-\Omega_2H(z,c_1)}{H(z,c_1)(w-c_1)}. \tag{3}$$

Because w is a meromorphic solutions of (2), we know that the zeroes of $w-c_1$ with multiplicity τ_1 are the poles of $\varphi_1(z;c_1)$ with multiplicity at most τ_1-1 by (3).

We take $c_1,c_2\in E,c_1\neq c_2$ and set

$$\begin{aligned}\varphi_2(z;c_1,c_2)&=\frac{\Omega_1-\Omega_2H(z,c_1)}{H(z,c_1)(w-c_1)}-\frac{\Omega_1-\Omega_2H(z,c_2)}{H(z,c_2)(w-c_2)}\\&=\frac{\Omega_1[H(z,c_2)(w-c_2)-H(z,c_1)(w-c_1)]}{H(z,c_2)(w-c_2)H(z,c_1)(w-c_1)}-\frac{(c_1-c_2)\Omega_2H(z,c_1)H(z,c_2)}{H(z,c_2)(w-c_2)H(z,c_1)(w-c_1)}.\end{aligned}$$

When z_0 is a zero of $w-c_1$(or $w-c_2$) with multiplicity τ_1(or τ_2) being neither poles of $a_{(i)},b_{(j)}$ nor zeros and poles of $H(z,c_k)(k=1,2)$, we have

$$\begin{aligned}&\Omega_1[H(z,c_2)(w-c_2)-H(z,c_1)(w-c_1)]-(c_1-c_2)\Omega_2H(z,c_1)H(z,c_2)\\&=\Omega_1[H(z,c_2)(w-c_1+c_1-c_2)-H(z,c_1)(w-c_1)]-(c_1-c_2)\Omega_2H(z,c_1)H(z,c_2)\\&=\Omega_1[H(z,c_2)(c_1-c_2)]-(c_1-c_2)\Omega_2H(z,c_1)H(z,c_2)=0.\end{aligned}$$

It shows that they are poles of $\varphi_2(z;c_1,c_2)$ with multiplicity at most τ_k-1 when zeroes of $w-c_k$ with multiplicity τ_k being neither poles of $a_{(i)}, b_{(j)}$ nor zeros and poles of $H(z,c_k)(k=1,2)$.

In general, we take distinct $c_1,c_2,\ldots,c_k\in E$ and set

$$\begin{aligned}\varphi_k(z;c_1,\ldots,c_k)=&\varphi_{k-1}(z;c_1,\ldots,c_{k-1})-\varphi_{k-1}(z;c_1,\ldots,c_{k-2},c_k)\\ =&(\Omega_1Q_{k-1}(z,w)-\Omega_2Q_{k-2}(z,w))/(\prod_{j=1}^{k}H(z,c_j)(w-c_j)),\end{aligned}\tag{4}$$

where $Q_k(z,w)$ is a polynomial of degree $k-1$ in w, its coefficients are combination with $H_j(z)(j=1,2,\ldots,k)$. By induction, from (4), it is evident that they are poles of $\varphi_k(z;c_1,\ldots,c_k)$ with multiplicity at most τ_j-1 when zeroes of $w-c_j$ with multiplicity τ_j being neither poles of $a_{(i)}$ and $b_{(j)}$ nor zeros and poles of $H_j(z)$.

Next we prove that $\varphi_k\equiv 0$ if $w(z)$ is an admissible solution of the differential equation (2).

Suppose $\deg_w^{H(z,w)}=k\geq\Delta$ and $\varphi_k\not\equiv 0$, by the first fundamental Theorem of Nevanlinna, it follows that

$$\begin{aligned}T(r,w)=&T(r,w-c_k)+O(1)=T(r,\prod_{j=1}^{k}(w-c_j)/\prod_{j=1}^{k-1}(w-c_j))+o(1)\\ \leq&T(r,\varphi_k/\prod_{j=1}^{k-1}(w-c_j))+T(r,\varphi_k/\prod_{j=1}^{k}(w-c_j))+O(1).\end{aligned}\tag{5}$$

Now we estimate $T(r,\varphi_k/\prod_{j=1}^{k-1}(w-c_j))$ and $T(r,\varphi_k/\prod_{j=1}^{k}(w-c_j))$. Let $k\geq\lambda$. Then

$$\begin{aligned}&m(r,\frac{\varphi_k}{\prod_{j=1}^{k-1}(w-c_j)})\\ =&m(r,\frac{\Omega_1Q_{k-1}(z,w)-\Omega_2Q_{k-2}(z,w)}{\prod_{j=1}^{k}H(z,c_j)(w-c_j)\prod_{j=1}^{k-1}(w-c_j)})\\ \leq&m(r,\frac{\Omega_1Q_{k-1}(z,w)}{\prod_{j=1}^{k}H(z,c_j)(w-c_j)\prod_{j=1}^{k-1}(w-c_j)})+m(r,\frac{\Omega_2Q_{k-2}(z,w)}{\prod_{j=1}^{k}H(z,c_j)(w-c_j)\prod_{j=1}^{k-1}(w-c_j)})+O(1)\\ \leq&m(r,\frac{\Omega_1}{\prod_{j=1}^{k}(w-c_j)})+m(r,\frac{Q_{k-1}(z,w)}{\prod_{j=1}^{k-1}(w-c_j)})+m(r,\frac{\Omega_2}{\prod_{j=1}^{k}(w-c_j)})+\\ &m(r,\frac{Q_{k-2}(z,w)}{\prod_{j=1}^{k-1}(w-c_j)})+2\sum m(r,\frac{1}{H(z,c_j)})+O(1).\end{aligned}$$

We note that

$$|\frac{w}{w-c_j}|\leq 1+\frac{|c_j|}{|w-c_j|}\leq(1+|c_j|)(\frac{1}{|w-c_j|})^{+}\leq c(\frac{1}{|w-c_j|})^{+},\tag{6}$$

where $|a|^+ = \max\{1, |a|\}, c = \max\{1 + |c_j|\}$. Thus

$$|\Omega_1 / \prod_{j=1}^{k} (w - c_j)| \le c^k \sum |a_{(i)}(z)| (\prod_j |\frac{w'}{(w - c_j)}|) \dots (\prod_j |\frac{w^{(n)}}{(w - c_j)}|),$$

$$|\Omega_2 / \prod_{j=1}^{k} (w - c_j)| \le c^k \sum |b_{(j)}(z)| (\prod_j |\frac{w'}{(w - c_j)}|) \dots (\prod_j |\frac{w^{(n)}}{(w - c_j)}|),$$

where $\prod_j |\frac{w^{(\alpha)}}{(w-c_j)}|$ is product of $i_{1\alpha}$ factors, $\prod_j (|\frac{1}{w-c_j}|)^+$ is product of $k - \lambda_t - t_0 (t = i, j)$ factors. So that

$$m(r, \Omega_1 / \prod_{j=1}^{k} (w - c_j)) \le \sum_{j=1}^{k} m(r, \frac{1}{w - c_j}) + \sum_{(i)} m(r, a_{(i)}) + O\{\sum \sum m(r, \frac{w^{(\alpha)}}{w - c_j})\}. \quad (7)$$

$$m(r, \Omega_2 / \prod_{j=1}^{k} (w - c_j)) \le \sum_{j=1}^{k} m(r, \frac{1}{w - c_j}) + \sum_{(j)} m(r, b_{(j)}) + O\{\sum \sum m(r, \frac{w^{(\alpha)}}{w - c_j})\}. \quad (8)$$

$$m(r, \frac{Q_{k-1}(z, w)}{\prod_{j=1}^{k-1} (w - c_j)}) \le \sum_{j=1}^{k-1} m(r, \frac{1}{w - c_j}) + \sum_{j=1}^{k-1} m(r, H_j) + O(1). \quad (9)$$

$$m(r, \frac{Q_{k-2}(z, w)}{\prod_{j=1}^{k-1} (w - c_j)}) \le \sum_{j=1}^{k-1} m(r, \frac{1}{w - c_j}) + \sum_{j=1}^{k-1} m(r, H_j) + O(1). \quad (10)$$

By (7),(8),(9),(10) and logarithmic derivative lemma,we have

$$m(r, \varphi_k / \prod_{j=1}^{k-1} (w - c_j)) \le 4 \sum_{j=1}^{k} m(r, \frac{1}{w - c_j}) + \sum_{(i)} m(r, a_{(i)}) + \sum_{(j)} m(r, b_{(j)}) + 2 \sum m(r, H_j) + S(r, w), \quad (11)$$

where $S(r, w) = O\{\log(rT(r, w))\}$.

Similarly we may deduce that

$$m(r, \varphi_k / \prod_{j=1}^{k} (w - c_j)) \le 4 \sum_{j=1}^{k} m(r, \frac{1}{w - c_j}) + \sum_{(i)} m(r, a_{(i)}) + \sum_{(j)} m(r, b_{(j)}) + 2 \sum m(r, H_j) + S(r, w). \quad (12)$$

Now we estimate $N(r, \varphi_k / \prod_{j=1}^{k-1} (w - c_j))$ and $N(r, \varphi_k / \prod_{j=1}^{k} (w - c_j))$. By

$$\frac{\varphi_k}{\prod_{j=1}^{k-1} (w - c_j)} = \frac{\Omega_1 Q_{k-1}(z, w) - \Omega_2 Q_{k-2}(z, w)}{\prod_{j=1}^{k} H(z, c_j)(w - c_j) \prod_{j=1}^{k-1} (w - c_j)}, \quad (13)$$

we know that the poles of $\varphi_k/\prod_{j=1}^{k-1}(w-c_j)$ may arise from the following cases:

(i) the poles of $\{a_{(i)}(z)\}, \{b_{(j)}(z)\}$.

(ii) the poles and the zeros of $\{H_j(z)\}$.

(iii) the zeroes of $w-c_j$ for which the cases (i) and (ii) are not satisfied.

(iv) the poles of $w(z)$.

For case (i), the contribution to $N(r, \varphi_k/\prod_{j=1}^{k-1}(w-c_j))$ is $\sum N(r, a_{(i)}) + \sum N(r, b_{(j)})$.

For case (ii), the contribution to $N(r, \varphi_k/\prod\limits_{j=1}^{k-1}(w-c_j))$ is $\sum N(r, H_j) + \sum N(r, \frac{1}{H_j})$.

For case (iii), the according to the above discussion,each zero with multiplicity τ_j is the poles of $\varphi_k/\prod\limits_{j=1}^{k-1}(w-c_j)$ with multiplicity at most $2\tau_j - 1$. Thus, the contribution is at most $\sum_{j=1}^{k-1}[2N(r, \frac{1}{w-c_j}) - \overline{N}(r, \frac{1}{w-c_j})]$.

For case (iv), if z_0 is a pole of w with multiplicity τ, then it is the poles of the denominator of right-side of the equality (13) with multiplicity $(2\Delta-1)\tau$, but z_0 is at most the poles of the numerator of right-side of the equality (13) with multiplicity $(2\Delta-1)\tau$. Hence, it follows that the poles of $w(z)$ doesn't arise from the poles of $\varphi_k/\prod_{j=1}^{k-1}(w-c_j)$.

Form the cases (i)-(iv), it yields that

$$\begin{aligned} N(r, \varphi_k/\prod_{j=1}^{k-1}(w-c_j)) \leq & \sum_{j=1}^{k-1}[2N(r, \frac{1}{w-c_j}) - \overline{N}(r, \frac{1}{w-c_j})] + \sum_{j=1}^{k-1} N(r, H_j) + \\ & \sum_{j=1}^{k-1} N(r, \frac{1}{H_j}) + \sum_{(i)} N(r, a_{(i)}) + \sum_{(j)} N(r, b_{(j)}). \end{aligned} \tag{14}$$

In a Similar fashion, we have

$$\begin{aligned} N(r, \varphi_k/\prod_{j=1}^{k}(w-c_j)) \leq & \sum_{j=1}^{k}[2N(r, \frac{1}{w-c_j}) - \overline{N}(r, \frac{1}{w-c_j})] + \sum_{j=1}^{k} N(r, H_j) + \\ & \sum_{j=1}^{k} N(r, \frac{1}{H_j}) + \sum_{(i)} N(r, a_{(i)}) + \sum_{(j)} N(r, b_{(j)}). \end{aligned} \tag{15}$$

Combining (5),(11),(12),(14) and (15), we obtain

$$\begin{aligned} T(r, w) \leq & 8\sum_{j=1}^{k} m(r, \frac{1}{w-c_j}) + \sum_{j=1}^{k}[4N(r, \frac{1}{w-c_j}) - 2\overline{N}(r, \frac{1}{w-c_j})] + 2\sum_{j=1}^{k} T(r, H_j) + \\ & 2\sum_{j=1}^{k} T(r, \frac{1}{H_j}) + 2\sum_{(i)} T(r, a_{(i)}) + 2\sum_{(j)} T(r, b_{(j)}) + S(r, w). \end{aligned} \tag{16}$$

We choose 17 systems distinct each other $\{c_j\}(j = 1, 2, \ldots, 17k)$ and apply the inequality

(16) to every system, combining the above seventeen inequlities,we deduce

$$17T(r,w) \le 8\sum_{j=1}^{17k} m(r,\frac{1}{w-c_j}) + \sum_{j=1}^{17k}[4N(r,\frac{1}{w-c_j}) - 2\overline{N}(r,\frac{1}{w-c_j})]+$$

$$2\sum_{j=1}^{17k} T(r,H_j) + 2\sum_{j=1}^{17k} T(r,\frac{1}{H_j}) + 34\sum_{(i)} T(r,a_{(i)}) + 34\sum_{(j)} T(r,b_{(j)}) + S(r,w).$$

By the second fundamental theorem of Nevanlinna,we have

$$17T(r,w) \le 16T(r,w) + 2\sum_{j=1}^{17k} T(r,H_j)+$$

$$2\sum_{j=1}^{17k)} T(r,\frac{1}{H_j}) + 34\sum_{(i)} T(r,a_{(i)}) + 34\sum_{(j)} T(r,b_{(j)}) + S(r,w),$$

i.e.,

$$T(r,w) \le 4\sum_{j=1}^{17k} N(r,H_j) + 4\sum_{j=1}^{17k} N(r,\frac{1}{H_j}) + 17\sum_{(i)} T(r,a_{(i)}) + 17\sum_{(j)} T(r,b_{(j)}) + S(r,w). \quad (17)$$

Because w is an admissible solution, by the inequality (17), we deduce $1 \le 0$. This is a contradiction. It follows that $\varphi_k \equiv 0$

This proves Theorem 1.

References:

[1] HE Yu-zan, XIAO Xiu-zhi. *Algebroid Functions and Ordinary Differential Equations* [M]. Beijing:Science Press, 1988.

[2] STEINMETZ N. *Eigensdhaften Eindeutiger Losungen Gewohnlicher Differentialgleichungen im Komplexen* [M]. Karlsurhe, Dissertation, 1978.

一类高阶代数微分方程的亚纯允许解

高 凌 云

(暨南大学数学系，广东 广州 510632)

摘 要: 利用亚纯函数的 Nevanlinna 值分布理论，研究了一类高阶微分方程具有亚纯允许解时，它所具有的形式，得到了一个 Malmquist 型定理.

关键词: 亚纯允许解；代数微分方程；有限聚点.