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On the Generation of Morse Lemma *

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Abstract: We find that: the Theorems 1 and 2, called "the generalization of Morse Lemma" in [1], are just a copy of a well known result in Singularity Theory, and the conclusion of Theorem 2 is not true. Since the Morse Lemma is an important result in Singularity Theory, we will discuss to the author of [1] for the problem of generalizing Morse Lemma.

Key words: Morse Lemma; Hessain; Decomposition Lemma.

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We want to discuss following problems to the author of [1].

Problem 1 The definitions 1 and 2 in [1] are not appropriate.
We cite two theorems in [2] as follows.

Theorem 1[2] *A cubic form of two variable may be reduced by a* $C-$*linear transformation to one of the forms:*

$$(1)\quad x^2y+y^3,\quad (2)\quad x^2y,\quad (3)\quad x^3,\quad (4)\quad 0$$

(in the real case: $x^2y \pm y^3$*).*

Theorem 2[2] *A function with initial cubic form* $x^2y \pm y^3$ *is equivalent to its initial form.*

The Theorem 1 of [1] is just a copy of Theorem 2[2] above. In the Definition 1 (and 2) of [1], the meaning that a germ $f \in M^3$ has non-degenerate cubic Hessain should be that j^3f has the forms: $(a_1x+b_1y)(a_2x+b_2y)(a_3x+b_3y)$, where three lines (real or complex) are different, by Theorem 1 and 2[2], j^3f is equivalent to $x^2y \pm y^3$. So, in fact, Theorem 1 of [1] does not make any new information other than Theorem 2[2].

Problem 2 Lemma 5 of [1] is a simple result of Linear Algebra, the detail proof of which given in [1] is not needed. Unfortunately, the conclusion of Theorem 2 is not true.

The following example is cited from [3](page 239):

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Consider, as an illustrative example, the Whitney family

$$W_t(x,y) = xy(x-y)(x-ty), \quad (x,y,t) \in R^3.$$

If restrict the parameter t to the interval $(1,\infty)$ then W_t is non-degenerate form for each t (Here "non-degenerate" isn't the same meaning with [1]). In particular, $W_t = 0$ consists of four distinct lines.

Intuitively, W_t and $W_{t'}$ are very much similar; yet, there does not exist a local $\mathcal{C}^1$-diffeomorphism h such that $W_{t'} \circ h = W_t$.(This can be proved using a simple linear Algebra argument on dh)

In addition, it is known that

$$N_t = xy(x-ty),$$

$$W_t = xy(x-y)(x-ty).$$

To N_t, by Theorem 2[2], N_t and $N_{t'}$ are $\mathcal{C}^\infty$-equivalent for any $t,\ t' \in (1,\infty)$. But the case for W_t is different. We know from [3] that W_t and $W_{t'}$ are $\mathcal{C}^0$-equivalent. Generally, we have the following result:

Theorem 1[4] *Let $Z = Z(x_1,\cdots,x_n)$ be any $\mathcal{C}^\infty$ function. Suppose there exist two positive numbers ε, δ such that*

$$|\mathrm{Grad}Z| \geq \varepsilon|X|^{r-\delta} \tag{*}$$

or equivalently,

$$|Z_{x_1}| + \cdots + |Z_{x_n}| \geq \varepsilon|X|^{r-\delta}$$

for all X near 0. Then $j^{(r)}(Z)$ is a $\mathcal{C}^0$-sufficient jet in $J^r(n,1)$.

The $(*)$ is called Lojasiewicz inequality, an important condition for determining the sufficiency of germs. If it holds we can find a vector field, which generates a one-parameter family of homeomorphisms trivializing the realization family of $Z(X)$. The results of [3] showed that Lojasiewicz inequality draws the "non-degenerate property" of germs in the highest degree of topology.

Problem 3 When the Hessain of a germ $f \in E_n$ is degenerate, there are several important ways to study classifications of germs. One of them is R. Thom's method, which considers the corank of the Hessain and he gets the seven elementary catastrophes. The another way is the so called "Blowing up" method, which introduces a new equivalence relation from Algebra Geometry. The famous mathematician of Chinese descent T.C. Kuo did many outstanding works in this field.

In [2] (pp.192-230) V.I. Arnold construct the apparatus of "quasi-homogeneous and semi-quasi-homogeneous" diffeomorphisms for reducing normal form quasi-homogeneous and semi-quasi-homogeneous singularities. One of his beautiful results is:

Theorem 3[2] *The quasi-homogeneous functions of two variables $m_0 = 0$ are given up to equivalence in the following exhaustive list:*

A_k	D_k	E_6	E_7	E_8
$x^{k+1}+y^2$	x^2y+y^{k-1}	x^3+y^4	x^3+xy^3	x^3+y^5

All non-degenerate functions with the same indices of quasi-homogeneity are reducible to one of the normal forms shown in the table.

Problem 4 One of generalization of Morse Lemma should be the Decomposition Lemma, which is the so called corner stone of Singularity Theory:

Decomposition Lemma ([4],p_{45}) Every germ f of corank q is isomorphic in E_n to a germ of the form

$$Q(x_1,x_2,\cdots,x_p)+g(x_{p+1},\cdots,x_n) \quad (p+q=n),$$

where the corank q is the corank of Hessain of f. Q is a quadratic form of maximal rank p in the variables $x_1,x_2,\cdots,x_p$, and $g\in M^3$ is a function only of the variables $x_{p+1},\cdots,x_{p+q}$.

Theorem 2[2] above can be gotten from this Lemma.

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关于 Morse 引理的推广

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摘 要: Morse Lemma 是奇点理论中一个极为重要的结论. [1] 的作者称其文中的定理 1 和定理 2 是 Morse Lemma 的推广. 为此我们愿就 [1] 中的几个问题与 [1] 的作者商榷.

关键词: Morse 引理; Hessain; 分裂引理.