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A Totally Umbilical Condition of Compact Space-like Hypersurfaces in the de Sitter Space *

WEN Bo[1], *WANG Qiang*[1], *PENG Cui-ying*[2]
(1. Dept. of Math., Changchun Taxation College, Jilin 130021, China;
2. Dept. of Math., Qingdao Scientific University, Shandong 266042, China)

Abstract: In this paper,an intrinsic condition for a Compact Space-like hypersurface with constant scalar curvature in a de Sitter space to be totally umbilical is obtained.

Key words: space-like hypersurface; sectional curvature; scalar curvature.

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1. Introduction

Let $M_S^{n+1}(c)$ be a $(n+1)$-dimensional connected Semi-Riemannian manifold of constant curvature c with index s. It is called an indefinite space form of index s and simply a space form when s=0. If $c > 0$, we call $M_1^{n+1}(c)$ a de Sitter space of index 1. Akutagawa[1] and Ramanathan[6] investigated space-like hypersurface in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Later, Cheng[2] generalized this result to general submanifolds in a de Sitter space.

To our knowledge, there are almost no intrinsic rigidity results for the space-like hypersurfsces with constant scalar curvature in a de Siteer space until Zheng[8] obtained the following result.

Theorem 1 *Let M be an n-dimensional compact space-like hypersurface in $M_1^{n+1}(c)$ with constant scalar curvature. If M satisfies*

(1) $K(M) > 0$,

(2) $\mathrm{Ric}(M) \leq (n-1)c$,

(3) $R < c$, *where R is the normalized scalar curvature of M,*

then M is totally umbilical.

In this paper, we will prove the following rigidity theorem for compact space-like hypersurface with constant scalar curvature in a Sitter space.

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Biography: WEN Bo (1950-), Associate Professor.

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Theorem 2 *Let M be an n-dimensional ($n \geq 3$) compact space-like hypersurface with constant normalized scalar curvature R in $M_1^{n-1}(c)$. If*

(1) $R - c \leq 0$, and

(2) the norm square $|B^2|$ of the second functional form of M satisfies

$$(n-R)c \leq |B|^2 < \frac{n[n(n-1)(c-R)^2 + 4(n-1)(c-R)c + nc^2]}{(n-2)[n(c-R)+2c]}, \tag{1}$$

then

$$|B|^2 \equiv n(c-R), \tag{2}$$

and m is totally umbilical.

2. Preliminaries

Let $M_1^{n+1}(c)$ be a $(n+1)$-dimensional semi-Riemannian manifold of constant curvature c with index 1. Let M be an n-dimensional Riemannian manifold immersed in $M_1^{n+1}(c)$.As the semi-Riemannian metric of $M_1^{n+1}(c)$ induces the Riemannian metric of M, M is called a space-like hypersurface. We choose a local field of semi-Riemannian orthonormal frames $e_1, \cdots, e_n, e_{n+1}$ in $M_1^{n+1}(c)$ such that at each point of $M, e_1, e_2, e_3, \cdots, e_n$ span the tangent space of M and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+1; 1 \leq i, j, k, \cdots \leq n. \tag{3}$$

Let $w_1, \cdots, w_{n+1}$ be its dual frame field so that the semi-Riemannian metric of $M_1^{n+1}(c)$ is given by $ds^2 = \Sigma_i (w_i)^2 - (w_{n+1})^2 = \Sigma_A e_A (w_A)^2$, where $e_i = 1$ and $e_{n+1} = -1$. Then structure equations of $M_1^{n+1}(c)$ are given by

$$dw_A = -\sum_B e_B w_{AB} \wedge w_B, w_{AB} + w_{BA} = 0, \tag{4}$$

$$dw_{AB} = -\sum_C e_C w_{AC} \wedge w_{CB} + \frac{1}{2} \sum_{C,D} K_{ABCD} w_C \wedge w_D, \tag{5}$$

$$K_{ABCD} = c e_A e_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{6}$$

Restrict these form to M, we have

$$w_{n+1} = 0. \tag{7}$$

The Rimannian metric of M is written as $ds^2 = \sum_i (w_i)^2$. From Cantan's lemma we can write

$$w_{n+1,i} = \sum_j h_{ij} w_j, h_{ij} = h_{ji}. \tag{8}$$

From these formulas ,we obtain the structure equations of M

$$dw_i = \sum_j w_{ij} \wedge w_j, w_{ij} + w_{ji} = 0, \tag{9}$$

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$$dw_{ij} = -\sum_k w_{ik} \wedge w_{ki} + \frac{1}{2}\sum_{k,l} K_{ijkl} w_k \wedge w_l, \tag{10}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{11}$$

where R_{ijkl} are the components of the curvature tensor of M.

For indefinite Riemannian manifolds in detail,refere to 0'Neill[4].

The quadratic form $h = \sum_{ij} h_{ij} w_i \otimes w_j$ is the second fundamental form of M. From the Gauss equation (11) we have

$$n(n-1)(c-R) = n^2H^2 - |B|^2, \tag{12}$$

where R is the normalized scalar curvature, $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature and $|B|^2$- $\Sigma ij h_{ij}$the norm square of the second fundamental form of M.

A hypersurface M is said to be totally umbilical if $h_{ij} = H\delta_{ij}$.

Codazzi equation is

$$h_{ijk} = h_{ikj}, \tag{13}$$

where the covariant derivative of the second fundamental form is defined by

$$\sum_k h_{ijk} w_k = dh_{ij} - \sum_k h_{ik} w_{kj} - \sum_k h_{jk} w_{ki}. \tag{14}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_i h_{ijkl} w_l = dh_{ijk} - \sum_i h_{ijl} w_{lk} - \sum_i h_{ilk} w_{lj} - \sum_i h_{ljk} w_{li}, \tag{15}$$

Then we have the following Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{jm} R_{mikl}. \tag{16}$$

For a C^2-function f defined on M, the fradient and the hessian (f_{ij}) are defined by

$$df = \sum_i f_i w_i, \sum_j f_{ij} w_j = df_i + \sum_j f_j w_{ji}. \tag{17}$$

The Laplacian of f is defined by $\triangle f = \sum_i f_{ii}$.

Let $T = \sum_{ij} T_{ij} w_i \otimes w_j$ be a symmetric tensor defined on M, where

$$T_{ij} = nH\delta_{ij} - h_{ij}. \tag{18}$$

Following Cheng and Yau[3], we introduce an operator associated to T acting on any C^2-function f by

$$\Box f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{19}$$

Since T_{ij} is divergence-free, it follows [3] that the operator is self-adjoint relaive to the L^2-inner product of M, i.e.,

$$\int_M f\Box g = \int g\Box f. \tag{20}$$

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Near a given point $p \in M$, we choose an orthonormal frame field $e_1, \cdots, e_n$ and their dual frame field $w_1, \cdots, w_n$ so that $h_{ij} = k_i\delta_{ij}$ at p. From (19)and (12) we have

$$\Box(nH) = nH \triangle (nH)_i - \sum (nH)_{ii} = \frac{1}{2} \triangle (nH)_i - \sum (nH)_{ii}^2 - \sum k_i (nH)_{ii}$$
$$= -\frac{1}{2} n(n-1) \triangle R + \frac{1}{2} \triangle |B|^2 - n^2 |\nabla H_i|^2 - \sum k_i (nH)_{ii}. \tag{21}$$

On the other hand, using(13) and (16), by a standard calculation we have

$$\frac{1}{2} \triangle |B|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i (nH)_{ii} + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2. \tag{22}$$

Substituting (22) into (21), we have

$$\Box(nH) = -\frac{1}{2} n(n-1) \triangle R + |\nabla B|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2. \tag{23}$$

3 . Lemmas and estimates

If the normalized scalar curvature R of M is constant, then from (23) we have

$$\Box(nH) = |\nabla B|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (k_i - k_j)^2. \tag{24}$$

From(11), we have $R_{ijij} = c - k_i k_j$. Substituting it into (24), we get

$$\Box(nH) - |\nabla B|^2 - n^2 |\nabla H|^2 - n^2 H^2 c + nc|B|^2 + |B|^4 - nH \sum_i k_i^3. \tag{25}$$

Let $\mu_i = k_i - H$ and $|Z|^2 = \sum_i \mu_i^2$. We have

$$\sum_i \mu_i = 0, |Z|^2 = |B|^2 - nH^2, \tag{26}$$

$$\sum_i k_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{27}$$

From (25)-(27), we obtain

$$\Box(nH) = |B|^2 - n^2 |\nabla H|^2 + |Z|^2 (nc - nH^2 + |Z|^2) - nH \sum_i \mu_i^3. \tag{28}$$

Lemma 3.1[5] *The same notations as above, for $n \geq 3$, we have*

$$-\frac{n-2}{\sqrt{n(n-1)}} |Z|^4 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} |Z|^3, \tag{29}$$

and the equality holds in (29) if and only if at least $(n-1)$ of the μ_i are equal.

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Combining (28) and (29), we have

$$\square(nH) \geq |B|^2 - n^2|\nabla H|^2 + |Z|^2(nc - nH^2 + |Z|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||Z|). \tag{30}$$

Lemma 3.2[3] *Assume the normalized scalar curvature R is a constant and $R - c \leq 0$. Then*

$$|\nabla B|^2 \geq n^2|\nabla H|^2. \tag{31}$$

Proof From (12), we have

$$n^2H^2 - \sum_{i,j} h_{ij}^2 = n(n-1)(c-R). \tag{32}$$

Taking the covariant derivative of the above equation, we obtain

$$n^2HH_k = \sum_{i,j} h_{ij}h_{ijk}. \tag{33}$$

It follows that

$$\sum_k n^4H^2(H_k)^2 = \sum_k(\sum_{i,j} h_{ij}h_{ijk})^2 \leq (\sum_{i,j} h_{ij}^2)\sum_{i,j,k} h_{ijk}^2, \tag{34}$$

that is

$$n^4H^2|\nabla H|^2 \leq |B|^2|\nabla B|^2. \tag{35}$$

On the other hand, from $c - R \geq 0$, we have $n^2H^2 - |B|^2 \geq 0$. Thus

$$n^2H^2|\nabla H|^2 \leq |H|^2|\nabla B|^2, \tag{36}$$

and Lemma 3.2 follows.

By (12), we know

$$|Z|^2 = |B|^2 - nH^2 = \frac{n-1}{n}(|B|^2 - n(c-R)). \tag{37}$$

Note that $|B|^2 \geq n(c-R)$, and $|B|^2 \equiv n(c-R)$ if and only if M is totally umbilical.

From (12), (30), (37) and Lemma 3.2, we get

$$\begin{aligned}\square(nH) =& \frac{n-1}{n}[|B|^2 - n(c-R)][nc - 2nH^2 + |B|^2 - (n-2)|H|\sqrt{|B| - n(c-R)}] \\ =& \frac{n-1}{n}[|B|^2 - n(c-R)][nc - 2(n-1)(c-R) + \frac{n-2}{n}|B|^2 - \\ & \frac{n-2}{n}\sqrt{(n(n-1)(c-R) + |B|^2)(|B| - n(c-R))}\ ].\end{aligned} \tag{38}$$

4. Proof of Theorem 2

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It is easy to check that our assumption condition (1), i.e.,

$$|B|^2 < \frac{n[n(n-1)(c-R)^2 + 4(n-1)(c-R)c + nc^2]}{(n-2)[n(c-R)+2c]}, \tag{39}$$

is equivalent to

$$[nc-2(n-1)(c-R)+\frac{n-2}{n}|B|^2]^2 > (\frac{(n-2)}{n})^2[n(n-1)(c-R)+|B|^2][|B|-n(c-R)]. \tag{40}$$

But it is clear from (39) that (40) is equivalent to

$$nc-2(n-1)(c-R)+\frac{n-2}{n}|B|^2 > (\frac{n-2}{n})\sqrt{(n(n-1)(c-R)+|B|^2)(|B|-n(c-R))}. \tag{41}$$

Therefore the right hand of (38) is non-negative. We also have $\int_M \Box(nH)dv = 0$, since M is compact and the operator is self-adjoint. Thus

$$|B|^2 \equiv n(c-R), \tag{42}$$

that is, $|B| = nH^2, M$ is totally umbilical. Thus, we complete the proof of Theorem 2.

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de Siteer 空间中拟紧致超曲面的一个全脐条件

温 波[1], 王 强[1], 彭 翠 英[2]

(1. 长春税务学院基础部, 吉林 长春 130021; 2. 青岛科技大学数学系, 山东 青岛 266042)

摘 要: 在这篇文章中，讨论了具有常数量曲率的拟紧致超曲面，并给出了它是全脐子流形的一个全脐条件。

关键词: 拟超曲面；截面曲率；数量曲率.

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