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Homologies of Infinite Quivers *

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Abstract: For a path algebra $A = kQ$ with Q an arbitrary quiver, consider the Hochschild homology groups $H_n(A)$ and the homology groups $\mathrm{Tor}_n^{A^e}(A, A)$, where A^e is the enveloping algebra of A. In this paper the groups are explicitly given.

Key words: Hochschild homology; infinite quiver.

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1. Introduction

The theory of Hochschild (co)homology has been used in several branches of mathematics. It is well-known that it has a close relation with the representation theory of finite-dimensional algebras.

Given a quiver Q and a field k, consider the path algebra $A = kQ$. It is clear that A is finite-dimensional if and only if Q is finite and has no oriented cycles. The algebra A, as well as its admissible quotient, plays an important role in the representation theory of finite-dimensional algebras.

On the other hand, in recent years, infinite-dimensional algebras and infinite-dimensional modules have aroused more and more interest. Quite naturally, we obtain an infinite-dimensional path algebra $A = kQ$ when Q is infinite, or contains oriented cycles. Note that A has a unit if and only if Q has only finitely many vertices.

We should be aware that in standard literatures (see e.g. [1], [2]), the Hochschild (co)homology was defined for associative algebras with unit. In this case, Cartan-Eilenberg had an important observation that the Hochschild cohomology group $H^n(A)$ coincides with the cohomology group $\mathrm{Ext}^n_{A^e}(A, A)$, and the Hochschild homology group $H_n(A)$ with the homology group $\mathrm{Tor}^n_{A^e}(A, A)$. This coincidence makes possible that some methods in representation theory provide useful information for computing these groups.

Unfortunately, this Cartan-Eilenberg coincidence fails for algebras without unit. This is the very reason why we have to deal with the Hochschild cohomology $H^n(A)$ and

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$\mathrm{Ext}^n_{A^e}(A,A)$ (resp. $H_n(A)$ and $\mathrm{Tor}^n_{A^e}(A,A)$) separately. In recent papers [3] and [4], P. Zhang proved that $\mathrm{Ext}^1_{A^e}(A,A)=0$ if and only if Q is a tree; and $H^1(A)=0$ if and only if Q is a finite tree; for a monomial algebra $\Lambda = kQ/I$ with Q connected, $H^1(\Lambda)=0$ if and only if Q is a finite tree.

The aim of this paper is to compute the Hochschild homology groups $H_n(A)$ and $\mathrm{Tor}^{A^e}_n(A,A)$ for $A=kQ$ with Q an arbitrary quiver. In particular, we prove that the following three statements are equivalent:

(i) $H_1(A)=0$.

(ii) $\mathrm{Tor}^{A^e}_1(A,A)=0$.

(iii) Q contains no oriented cycles.

This generalizes the corresponding result in [5].

Throughout, k is a field, and we denote $\otimes_k$ by $\otimes$.

2. Quivers

2.1. A quiver $Q=(Q_0,Q_1)$ is an oriented graph, where Q_0 is the set of vertices and Q_1 the set of arrows between vertices. We denote by $h: Q_1 \to Q_0$ and $t: Q_1\to Q_0$ the maps where $h(\alpha)=i$ and $t(\alpha)=j$ when $\alpha: i\to j$ is an arrow from the vertex i to the vertex j. A path in the quiver Q is either an ordered sequence of arrows $p=\alpha_1\cdots\ \alpha_n$ with $t(\alpha_s)=h(\alpha_{s+1})$ for $1\le s<n$, or the symbol e_i for $i\in Q_0$. We call the paths e_i trivial paths and we define $h(e_i)=t(e_i)=i$. For a nontrivial path $p=\alpha_1\cdots\ \alpha_n$ we define $h(p)=h(\alpha_1)$, $t(p)=t(\alpha_n)$ and $l(p)=n$, which are respectively called the head, the tail, and the length of p. A nontrivial path p is said to be an oriented cycle if $h(p)=t(p)$. An oriented cycle $p=\alpha_1\cdots\ \alpha_n$ is said to be basic if $h(\alpha_1)$, $h(\alpha_2)$, $\cdots$, $h(\alpha_n)$ are distinct from each other.

We emphasize that the quivers $Q=(Q_0,Q_1)$ considered in this paper are arbitrary, i.e., Q can be infinite, namely, at least one of Q_0 and Q_1 is an infinite set.

2.2. For a field k and a quiver Q, let $A=kQ$ be the k-vector space with the paths of Q as basis. For $p=\alpha_1\cdots\ \alpha_m$, $q=\beta_1\cdots\ \beta_n$, define the multiplication

$$pq=\begin{cases}\alpha_1\cdots\alpha_m\beta_1\cdots\beta_n, & t(p)=h(q),\\ 0, & t(p)\neq t(q).\end{cases}$$

In this way, $A=kQ$ becomes a k-algebra, which is called the path algebra of Q. Note that A has the unit if and only if Q_0 is a finite set, and in this case $1=\sum_{i\in Q_0} e_i$; and that A is finite-dimensional if and only if Q is a finite quiver (i.e., both Q_0 and Q_1 are finite sets) and Q contains no oriented cycles. As the title indicates, we are interested in infinite quivers.

Note that $A=kQ$ has a set of orthogonal idempotents $\{e_i | i\in Q_0\}$ and $A=\oplus_{i\in Q_0} e_iA$ $=\oplus_{i\in Q_0} Ae_i$. Consider the category A-Mod of all left A-modules X with $X=\oplus_{i\in Q_0} e_iX$. Clearly, Ae_i, $A\in A$-Mod; and A-Mod is an extension closed abelian category.

Note that most part of the standard results in homological algebra for a ring with unit is still valid in this category. In particular, we can consider the homology of A which is the main purpose of this article.

2.3. Recall that a ring R is said to be hereditary provided every submodule of a projective module is also projective. Here, we do not insist that R have a unit, but assume that R has a set of orthogonal idempotents $\{e_i | i \in I\}$ such that $R = \oplus_{i \in I} Re_i = \oplus_{i \in I} e_i R$. In the following, we will show that $A = kQ$ is hereditary where $Q = (Q_0, Q_1)$ is an arbitrary quiver. For $X \in A$-Mod, the following construction of a projective resolution of X is the explicit form of Happel's resolution in [6], which was stated for A being finite-dimensional. In [2], P. Zhang pointed out that the resolution still holds for infinite quivers.

Lemma 2.1 *We have the short exact sequence of A-modules*

$$0 \to \bigoplus_{\alpha \in Q_1} (Ae_{h(\alpha)} \otimes e_{t(\alpha)} X) \xrightarrow{f} \bigoplus_{i \in Q_0} (Ae_i \otimes e_i X) \xrightarrow{g} X \to 0, \tag{1}$$

where g and f are homomorphisms defined by

$$g(a \otimes x) = ax \quad \text{for} \quad a \in Ae_i \quad \text{and} \quad x \in e_i X;$$

$$f(a \otimes x) = a\alpha \otimes x - a \otimes \alpha x \quad \text{for} \quad a \in Ae_{h(\alpha)} \quad \text{and} \quad x \in e_{t(\alpha)} X.$$

For the proof, see [2]. The following corollary is a direct consequence of the lemma.

Corollary 2.2 *For $A = kQ$, we have the following*
(i) For $X \in A$-Mod, the projective dimension $p.d.X \leq 1$;
(ii) A is hereditary.

3. Homology groups $\mathrm{Tor}_n^{A^e}(A, A)$

3.1. Let $A = kQ$ with $Q = (Q_0, Q_1)$ a quiver. Consider $A = A \otimes A^*$, the enveloping algebra of A, where A^* is the opposite algebra of A. In A^e we have $(a \otimes b')(c \otimes d') = ac \otimes (db)'$ and $A^e = \oplus_{i,j \in Q_0} A^e(e_i \otimes e_j') = \oplus_{i,j \in Q_0} Ae_i \otimes (e_j A)'$. Any A-bimodule X can be regarded as a right A^e-module in a natural way: $x \cdot (a \otimes b') = bxa$ for $a \otimes b' \in A^e$, $x \in X$. In this section, we will consider the homology groups $\mathrm{Tor}_n^{A^e}(A, A)$. The following lemma will give an explicit form of projective resolution of A over A^e, and this is the key to compute homology groups. The proof is an easy consequence of Lemma 2.1.

Lemma 3.1 *We have the following projective resolution of A over A^e*

$$0 \to \bigoplus_{\alpha \in Q_1} (Ae_{h(\alpha)} \otimes e_{t(\alpha)} A)') \xrightarrow{f} \bigoplus_{i \in Q_0} (Ae_i \otimes e_i A)') \xrightarrow{g} A \to 0, \tag{2}$$

where g and f are A^e-homomorphisms defined by

$$g(a \otimes b') = ab \quad \text{for} \quad a \in Ae_i \quad \text{and} \quad b \in e_i A;$$

$$f(a \otimes b') = a\alpha \otimes b - a \otimes \alpha b \quad \text{for} \quad a \in Ae_{h(\alpha)} \quad \text{and} \quad b \in e_{t(\alpha)} A.$$

3.2. With the projective resolution in Lemma 3.1, we can now compute the homology groups $\mathrm{Tor}_n^{A^e}(A,A)$.

Theorem 3.2 *Let $A = kQ$ with Q an arbitrary quiver. Then*

(i) $\mathrm{Tor}_n^{A^e}(A,A) = 0$ *for* $n \geq 2$.

(ii) If Q contains oriented cycles, then $\mathrm{Tor}_1^{A^e}(A,A)$ is an infinite-dimensional k-vector space.

(iii) $\mathrm{Tor}_1^{A^e}(A,A) = 0$ *if and only if Q contains no oriented cycles.*

(iv) If Q contains no oriented cycles, then $\mathrm{Tor}_0^{A^e}(A,A) = k^q$, where $q = |Q_0|$; If Q contains oriented cycles, then $\mathrm{Tor}_0^{A^e}(A,A)$ is a infinite-dimensional k-vector space.

Proof The assertion (i) follows directly from Lemma 3.1.

In order to compute the homology groups, first apply $A \otimes_{A^e} -$ to the exact sequence (2) in Lemma 3.1. Then we have the following exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{A^e}(A,A) \longrightarrow \bigoplus_{\alpha \in Q_1} A \otimes_{A^e} (Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)')$$
$$\xrightarrow{1\otimes f} \bigoplus_{i \in Q_0} A \otimes_{A^e} (Ae_i \otimes (e_iA)') \xrightarrow{1\otimes g} A \otimes_{A^e} A \longrightarrow 0. \qquad (3)$$

Now let Q be a quiver that contains oriented cycles. In the following we will show that $\ker(1 \otimes f)$ is an infinite-dimensional k-vector space. Without loss of generality, let a basic cycle $Q' = (Q'_0, Q'_1)$ of length m be a subquiver of Q, where $Q'_0 = \{1, 2, \cdots, m\}$, $Q'_1 = \{\alpha_i : i \to i+1, \text{ for } i = 1, \cdots, m-1;\ \alpha_m : m \to 1\}$. Denote $p_1 = \alpha_2 \cdots \alpha_m$, $p_2 = \alpha_3 \cdots \alpha_m \alpha_1$, $\cdots$, $p_m = \alpha_1 \cdots \alpha_{m-1}$ and $c_i = \alpha_i p_i$ for $i = 1, 2, \cdots, m$. For integer $t \geq 0$, we have

$$\begin{aligned}
&(1\otimes f)\Big(\sum_{i=1}^{m-1} p_i c_i^t \otimes_{A^e} (e_i \otimes e'_{i+1}) + p_m c_m^t \otimes_{A^e} (e_m \otimes e'_1)\Big)\\
&= \sum_{i=1}^{m-1} p_i c_i^t \otimes_{A^e} (\alpha_i \otimes e_{i+1} - e_i \otimes \alpha_i) + p_m c_m^t \otimes_{A^e} (\alpha_m \otimes e_1 - e_m \otimes \alpha_m)\\
&= \sum_{i=1}^{m-1} (c_{i+1}^{t+1} - c_i^{t+1}) \otimes_{A^e} (e_i \otimes e_{i+1}) + (c_1^{t+1} - c_m^{t+1}) \otimes_{A^e} (e_m \otimes e_1)\\
&= \Big[\sum_{i=1}^{m-1} (c_{i+1}^{t+1} - c_i^{t+1}) + (c_1^{t+1} - c_m^{t+1})\Big] \otimes_{A^e} \Big(\sum_{i=1}^{m-1} e_i \otimes e_{i+1} + e_m \otimes e_1\Big)\\
&= 0.
\end{aligned}$$

This shows that $\ker(1 \otimes f)$ contains the linearly independent infinite set

$$\Big\{\sum_{i=1}^{m-1} p_i c_i^t \otimes_{A^e} (e_i \otimes e'_{i+1}) + p_m c_m^t \otimes_{A^e} (e_m \otimes e'_1)),\ t \geq 0\Big\}.$$

This proves the assertion (ii) since by (3) we have $\mathrm{Tor}_1^{A^e}(A,A) = \ker(1 \otimes f)$.

For (iii), it remains to prove $\mathrm{Tor}_1^{A^e}(A,A)=0$ when Q has no oriented cycles. Note that when Q has no oriented cycles, we have

$$\begin{aligned} A\otimes_{A^e}(Ae_{h(\alpha)}\otimes(e_{t(\alpha)}A)') &= A\otimes_{A^e}A^e(e_{h(\alpha)}\otimes e_{t'(\alpha)}) \\ &= A(e_{h(\alpha)}\otimes e_{t'(\alpha)})\otimes_{A^e}(e_{h(\alpha)}\otimes e_{t'(\alpha)}) \\ &= e_{t(\alpha)}Ae_{h(\alpha)}\otimes_{A^e}(e_{h(\alpha)}\otimes e'_{t(\alpha)})=0 \end{aligned}$$

for any $\alpha\in Q_1$. It follows that $\mathrm{Tor}_1^{A^e}(A,A)=0$ by (3).

Finally, we prove (iv). First, by the definition of homology groups, we have $\mathrm{Tor}_0^{A^e}(A,A)=A\otimes_{A^e}A$. Note that $A=\bigoplus_{i,j\in Q_0}e_iAe_j=\bigoplus_{i,j\in Q_0}A(e_j\otimes e'_i)$, so we have

$$\begin{aligned} A\otimes_{A^e}A &= \bigoplus_{i,j\in Q_0}A(e_j\otimes e'_i)\otimes_{A^e}A=\bigoplus_{i,j\in Q_0}A(e_j\otimes e'_i)^2\otimes_{A^e}A \\ &= \bigoplus_{i,j\in Q_0}A(e_j\otimes e'_i)\otimes_{A^e}(e_j\otimes e'_i)A=\bigoplus_{i,j\in Q_0}e_iAe_j\otimes_{A^e}e_jAe_i. \end{aligned}$$

If Q has no oriented cycles, then for $i\neq j$, either $e_iAe_j=0$, or $e_jAe_i=0$; for $i=j$, $e_iAe_j=k$, as k-spaces. Hence, in this case $\mathrm{Tor}_0^{A^e}(A,A)=k^q$, where $q=|Q_0|$. If Q contains oriented cycles, then when i and j are vertices in the same cycles, both e_iAe_j and e_jAe_i are infinite-dimensional vector spaces. Now it is easy to know that in this case $\mathrm{Tor}_1^{A^e}(A,A)$ is an infinite-dimensional k-vector space (no matter $|Q_0|$ is infinite or not).

4. Hochschild homology

4.1. The Hochschild homology of a k-algebra Λ is the homology of the complex $(\Lambda^{\otimes n},\ d)$ with

$$d(a_1\otimes\cdots\otimes a_n)=a_2\otimes\cdots\otimes a_{n-1}\otimes a_na_1+\sum_{i=1}^{n-1}(-1)^i(a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n).\quad(4)$$

We denote the n-th homology group by $H_n(A)$, for $n\geq 0$. Note that in standard literatures (see e.g. [2]), the Hochschild (co)homology was defined for algebras with unit and in that case we have the following Cartan-Eilenberg identities:

Lemma 4.1 *For an algebra A with unit, $H_n(A)\cong\mathrm{Tor}_n^{A^e}(A,A)$ and $H^n(A)\cong\mathrm{Ext}_n^{A^e}(A,A)$ for any integer $n\geq 0$ as k-vector spaces.*

Here we have a remark that when the algebra A has no unit, Lemma 4.1 is no longer valid. For example, when Q is an infinite tree, $\mathrm{Ext}^1_{A^e}(A,A)=0$, but $H^1(A)$ is not zero. For detail, see [3] and [4]. For the case of homology groups, we conjecture that the identity also fails.

4.2. The Hochschild homology of an algebra whose quiver is finite and has no oriented cycles was explicitly given by Cibils in [5]. In this subsection we will show that the corresponding result is still valid when we make no restriction on finiteness.

Let $Q=(Q_0,Q_1)$ be a quiver, $A=kQ$ the path algebra and F the two-sided ideal generated by the arrows of Q. Let I be some two-sided ideal of kQ such that $I\subset F$.

Denote by Λ the quotient algebra A/I with Q an arbitrary quiver which contains no oriented cycles.

Theorem 4.2 *The Hochschild homology of Λ vanishes in positive degrees and is k^q in degree zero, where $q = |Q_0|$.*

To prove the theorem we need the following notations and facts. It is clear that we can choose B from the set of oriented paths in Q such that B is a basis of Λ. Then for each n, we have that B^n is a k-basis of $\Lambda^{\otimes n}$. From now on, we write the element $a = a_1 \otimes \cdots \otimes a_n \in \Lambda^{\otimes n}$ as $(a_1, \cdots, a_n)$. Let Λ_0^n be the k-subspace of $\Lambda^{\otimes n}$ with basis $\Delta Q_0^n = \{(x,\ x,\ \cdots,\ x) | x \in Q_0\}$, L^n the k-subspace with basis $D^n = B^n \backslash \Delta Q_0^n$. Now we have defined two subcomplexes of the Hochschild complex of Λ. Moreover, we have $(\Lambda^{\otimes n},\ d) = (\Lambda_0^n,\ d) \oplus (L^n,\ d)$. It is obvious that the complex $(\Lambda_0^n,\ d)$ is isomorphic to the direct sum of q copies of the Hochschild complex of k. It is easy to check that $H_0(k) = k$ and $H_i(k) = 0$ for $i > 0$. This together with the following lemma proves the theorem.

Lemma 4.3 *The complex $(L^n,\ d)$ is acyclic.*

Proof Let $a = (a_1,\ \cdots,\ a_n) \in D^n$. The composition is not an oriented cycle. This means exactly that $t(a_n) \neq h(a_1)$ (we say that a is of type I) or there exist $i \in \{1,\ \cdots,\ n-1\}$ such that $t(a_i) \neq h(a_{i+1})$ (a is of type II). The lemma is proved using an explicit homotopy contraction.

Type I. If $t(a_n) \neq h(a_1)$, we define $s(a_1,\ \cdots,\ a_n) = -(h(a_1),\ a_1,\ \cdots,\ a_n)$.

Type II. If $t(a_n) = h(a_1)$, let r be the smallest integer in $\{1,\ 2,\ \cdots,\ n-1\}$ such that $t(a_r) \neq h(a_{r+1})$. We define $s(a_1,\ \cdots,\ a_n) = (-1)^{r+1}(a_1, \cdots,\ a_r,\ h(a_{r+1}),\ a_{r+1},\ \cdots,\ a_n)$.

We want to check $sd + ds = 1$.

Let $a = (a_1, \cdots, a_n) \in D^n$ with $t(a_n) \neq h(a_1)$ (type I). Then

$$ds(a) = 0 + (a_1, \cdots, a_n) + \sum_{i=1}^{n-1}(-1)^i(h(a_1), a_1, \cdots, a_i a_{i+1}, \cdots, a_n),$$

$$d(a) = 0 + \sum_{i=1}^{n-1}(-1)^i(a_1, \cdots, a_i a_{i+1}, \cdots, a_n).$$

Let $a_i a_{i+1} = \sum_{c \in B} \lambda_c c$. We have

$$s(a_1, \cdots, a_i a_{i+1}, \cdots, a_n) = \sum_{c \in B} \lambda_c s(a_1, \cdots, c, \cdots, a_n).$$

Each summand is of type I. Then we have

$$s(a_1, \cdots, a_i a_{i+1}, \cdots, a_n) = -(h(a_1), a_1, \cdots, c, \cdots, a_n)$$

and $sd + ds = 1$.

Let $a = (a_1, \cdots, a_n) \in D^n$ be a vector of type II and suppose $r = 1$. Then

$$\begin{aligned} ds(a) &= d(a_1, h(a_2), a_2, \cdots, a_n) \\ &= (h(a_2), a_2, \cdots, a_n a_1) - 0 + (a_1, a_2, \cdots, a_n) + \\ &\quad \sum_{i=2}^{n-1}(-1)^{i+1}(a_1, h(a_2), a_2, \cdots, a_i a_{i+1}, \cdots, a_n). \end{aligned}$$

The first summand of $d(a)$ is of type I. The other summand are of type II with the same $r = 1$.

$$sd(a) = -(h(a_2), a_2, \cdots, a_n a_1) + 0 + \sum_{i=2}^{n-1} (-1)^i (a_1, h(a_2), a_2, \cdots, a_i a_{i+1}, \cdots, a_n).$$

Then $ds + sd = 1$ when a is of type II and $r = 1$.

Let $a = (a_1, \cdots, a_n) \in D^n$ be a vector of type II with $r \geq 2$. Then

$$\begin{aligned} ds(a) =& (-1)^{r+1}(a_2, \cdots, a_r, h(a_{r+1}), a_{r+1}, \cdots, a_n a_1) + \\ & \sum_{i=1}^{r-1} (-1)^{r+1}(-1)^i (a_1, \cdots, a_i a_{i+1}, \cdots, a_r, h(a_{r+1}), a_{r+1}, \cdots, a_n) + \\ & 0 + (-1)^{2(r+1)}(a_1, \cdots, a_n) + \\ & \sum_{i=r+1}^{n-1} (-1)^{r+1}(-1)^{i+1} (a_1, \cdots, a_r, h(a_{r+1}), a_{r+1}, \cdots, a_i a_{i+1}, \cdots, a_n). \end{aligned}$$

To compute sd note that each summand of $d(a)$ is of type II with $r \geq 2$. Then

$$\begin{aligned} sd(a) =& (-1)^r (a_2, \cdots, a_r, h(a_{r+1}), a_{r+1}, \cdots, a_n a_1) + \\ & \sum_{i=1}^{r-1} (-1)^r (-1)^i (a_1, \cdots, a_i a_{i+1}, \cdots, a_r, h(a_{r+1}), a_{r+1}, \cdots, a_n) + \\ & \sum_{i=r+1}^{n-1} (-1)^{r+1}(-1)^i (a_1, \cdots, a_r, h(a_{r+1}), a_{r+1}, \cdots, a_i a_{i+1}, \cdots, a_n). \end{aligned}$$

This proves that $sd + ds = 1$ when a is of type II with $r \geq 2$.

Now for all the basis vectors of L^n we obtain $sd + ds = 1$. Thus we have $sd + ds = 1$.

4.3. In this subsection, we will consider the Hochschild homology of a path algebra whose quiver may have oriented cycles. We need the following lemma, which is a simple case of Theorem 2 in [7] when Q is a finite quiver.

Lemma 4.4 *Let $A = kQ$ be a path algebra. Assume that Q' is a connected subquiver of Q and $B = kQ'$. Then $H_n(B)$ is a direct summand of $H_n(A)$.*

Proof Let Ω be a k-basis of A which consists of oriented paths in Q, and I be the ideal in A generated by $Q_0 \backslash Q_0'$ and $Q_1 \backslash Q_1'$. It is clear that Ω is the disjoint union of $B \cap \Omega$ and $I \cap \Omega$, so $A = B \oplus I$ as k-vector spaces. By definition, the Hochschild complex of A is $C(A) = (A^{\otimes n}, d)$, where d is defined as (4). Then for every $n \geq 0$, $C_n(A) = C_n(B) \oplus C_n(B, I)$, where $C_n(B, I)$ is a finite direct sum of some k-spaces of the form $V_1 \otimes \cdots \otimes V_n$ with $V_i = B$ or $V_i = I$ such that there is at least one $V_i = I$. Observe that $(B^{\otimes n}, d)$ is a subcomplex of $(A^{\otimes n}, d)$; since I is an ideal in A, we also see that d maps $C_n(B, I)$ into $C_{n-1}(B, I)$ by (4). So $C(A)$ is a direct sum of subcomplexes $C(B)$ and $(C_n(B, I), d)$, hence $H_n(B)$ is a direct summand of $H_n(A)$. □

Combining (ii) of Theorem 3.2, Lemma 4.1, Theorem 4.2, and Lemma 4.4, we obtain the following theorem which is the main result on Hochschild homology of a path algebra.

Theorem 4.5 *Let $A = kQ$ with Q an arbitrary quiver.*

(i) $H_n(A) = 0$ for $n \geq 1$ if Q contains no oriented cycles.

(ii) $H_1(A)$ is an infinite-dimensional k-vector space if Q contains oriented cycles.

4.4. We conclude with an interesting consequence of Theorem 3.2 and Theorem 4.5.

Theorem 4.6 *Let $A = kQ$ with Q an arbitrary quiver. Then the following three statements are equivalent.*

(i) $H_1(A) = 0$.

(ii) $\mathrm{Tor}_1^{A^e}(A, A) = 0$.

(iii) Q contains no oriented cycles.

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无限箭图的同调群

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摘 要: 对任意箭图 Q，我们研究路代数 $A = kQ$ 的 Hochschild 同调群 $H_n(A)$ 和同调群 $\mathrm{Tor}_n^{A^e}(A, A)$，其中 A^e 是代数 A 的包络代数. 在本文中，我们具体地给出了各次同调群和 Hochschild 同调群.

关键词: Hochschild 同调；无限箭图.