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On Chromatic Polynomials of Complements of All Wheels with Any Missing Consecutive Spokes *

HAO Rong-xia, LIU Yan-pei

(Dept. of Math., Northern Jiaotong University, Beijing 100044, China)

Abstract: In this paper, a new method has been used to calculate the chromatic polynomials of graphs. In particular, the chromatic polynomials of complements of all wheels with any missing consecutive spokes are given.

Key words: chromatic polynomial; wheels; complement.

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1. Introduction

Let G be a simple graphs with n vertices. A *k-independent partition* is a partition $\{A_1, A_2, \cdots A_k\}$ of the vertex set of G, where k is a positive integer, and each A_i is a nonempty independent set of G. Let $\alpha(G,k)$ be the number of k-independent partition of G. The chromatic polynomial [1] of G is given by

$$f(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)[\lambda]_k, \tag{1}$$

where $[\lambda]_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$.

If every component of spanning subgraph H of graph G is a complete graph, then H is called an ideal subgraph of G. Let $N(G,k)$ denote the number of ideal subgraph with k components of G. It is easy to see that $N(\hat{G},k) = \alpha(G,k)$, where $\hat{G}$ is the complement of G. Thus the chromatic polynomial can be written as

$$f(G,\lambda) = \sum_{k=1}^{n} N(\hat{G},k)[\lambda]_k. \tag{2}$$

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Biography: HAO Rong-xia (1965-), female, Ph.D., Associate Professor.

The adjoint polynomial[2] of G is defined to be the polynomial

$$h(G,\mu)=\sum_{k=1}^{n}N(G,k)\mu^{k}. \tag{3}$$

Let $h(G,\mu)$ be a polynomial and

$$h(G,\mu)=\mu^{\alpha(G)}h_1(G,\mu),$$

where $\mu^{\alpha(G)}$ is the minimal power of x with nonzero coefficient in $h(G,\mu)$. If $h_1(G,\mu)$ is an irreducible polynomial over the rational number field, then G is called an irreducible graph.

The wheel W_n with n vertices ($n\geq 2$) is the $(n-1)$-cycle with an additional vertex v which is adjacent to each of the $n-1$ vertices on the $(n-1)$-cycle. For convenient, let W_1 be the wheel with one vertex graph. The fan S_n with n vertices is defined as the $(n-1)$-path with an additional vertex v adjacent to each of the $(n-1)$ vertices on the $(n-1)$-path. For convenient, let S_1 be the fan which is the vertex graph. Let $W(n,k)$ be the graph obtained from W_n by deleting all but k consecutive spokes,where $1\leq k\leq n-1$.

Let P_n be the path with n vertices and K_n the complete graph with n vertices.

Although the chromatic polynomial $f(G;\lambda)$ of a graph G has been studied extensively since its introduction by Birkhoff in 1912, as it encodes quite a lot of information about the graph, there are few kinds of graphs of which chromatic polynomials can be calculated with formulas. For most graphs (specially for graphs with larger order), computing their chromatic polynomials is inconvenient. Chromaticity of the complements of paths and cycles is studied in [3] and in [2]. The computing formulas for chromatic polynomials of the complement $\hat{P}_n$ of a path with n vertices and the complement $\hat{C}_n$ of n-cycle C_n with n vertices are given in [4]. The chromaticity of $W(n,k)$ had been studied in [5] and [6] et al., but there is no any result about that of the complements of $W(n,k)$ for any positive k.

In present paper, by extending an algebraic function adjoint sum function to two parameters which is correspondence with chromatic sum [7],[8],[9] and applying this function to find the chromatic polynomials of complements of wheels with any consecutive spokes $W(n,k)$.

In general, let $\mathcal{M}$ be the set of some kind of graphs, *the adjoint sum function* of $\mathcal{M}$ is defined as $g_{\mathcal{M}}(x,\mu)=\sum_{G\in\mathcal{M}}h(G,\mu)x^{n(G)}$, where $n(G)$ is the order of graph G, *i.e.*, the number of vertices (in fact, $n(G)$ may be replaced by another parameter of G, or a number of parameters of G if necessary). The coefficient of x^n in $g_{\mathcal{M}}(x,\mu)$ is called adjoint sum.

The following Theorems1-4 had been obtained

Theorem 1[2] $f(\hat{P}_n,t)=\sum_{\frac{n}{2}\leq k\leq n}\binom{k}{n-k}[t]_k$, *where P_n is a path with n vertices.*

Theorem 2[2] $f(\hat{C}_n,t)=\sum_{\frac{n}{2}\leq k\leq n}\frac{n}{k}\binom{k}{n-k}[t]_k(n\geq 4)$, *where C_n is a cycle with n vertex.*

Theorem 3

$$f(\hat{S}_n,t)=\sum_{\lfloor\frac{n}{2}\rfloor\leq k\leq n}N(S_n,k)[t]_k \quad \textit{for} \quad n>3,$$

$$f(\hat{S}_n,t) = N(S_n,1)t + \sum_{2\le k\le 3} N(S_n,k)[t]_k \quad for \quad n \le 3,$$

where

$$N(S_n,1) = \frac{1}{(n-1)!(1-n)!} + \frac{1}{(n-2)!(3-n)!}, \tag{4}$$

$$N(S_n,r) = \frac{(r-1)![(2r-n+1)(2r-n)+r^2(n-r)]}{(n-r)!(2r-n+1)!} \tag{5}$$

for $r > 1$ and S_n is the fan with n vertices.

Theorem 4 $f(\hat{W}_n,t) = \sum_{\lfloor\frac{n}{2}\rfloor\le k\le n} N(W_n,k)[t]_k$ *for $n > 3$, where*

$$N(W_n,r) = \frac{(r-2)!(n-1)[r(n-r)(r-1)+(2r-n+1)(2r-n)]}{(n-r)!(2r-n+1)!}, \tag{6}$$

for $r \ge 2$ and W_n is the wheel with n vertices.

Since the wheel W_n is the special case of $W(n,k)$ when $k = n-1$, while the cycle C_n is relate to $W(n,k)$ when $k = 0$, thus the study of chromatic polynomial of $W(n,k)$ is the generalization of those of W_n and C_n

In present paper, The following chromatic polynomial of complement $\hat{W}(n,k)$ of wheels with any missing consecutive spokes $W(n,k)$ is proved.

Theorem 5 $f(\hat{W}_{(n,k)},t) = \sum_{\lfloor\frac{n}{2}\rfloor\le r\le n} N(W_{(n,k)},r)[t]_r$ *for $n > 3, k < n-1$, where*

$$N(W_{(n,k)},r) = N(G(n,k),r) + N(G(n-2,k-1),r-1), \tag{7}$$

and

$$N(G(n,k),m) = \sum_{i\ge 0}^{m-1} a_{nk}^i [iN(S_{k+1},m-i) + (2i-n+k+1)N(S_k,m-i)], \tag{8}$$

where $a_{nk}^i = \frac{(i-1)!}{(n-k-i-1)!(2i-n+k+1)!}$ and $N(S_r,l)$ is the same as Theorem 3.

2. Main results

Lemma 1[10] *If $G_1, G_2, \cdots, G_k$ are the components of G, then*

$$h(G,\mu) = \prod_{i=1}^{k} h(G_i,\mu). \tag{9}$$

For $x, y \in V(G)$, let $G \bullet xy$ be the graph obtained from G by identifying x and y and replacing multi-edges by single ones. For $xy \in E(G)$, let

$$E_1(xy) = \{xu \in E(G) | u \ne y, yu \notin E(G)\} \cup \{yv \in E(G) | v \ne x, xv \notin E(G)\}$$

for $x \ne y$, let $G \bullet xy$ be the graph $(G - E_1(xy)) \bullet xy$. By the definition of $N(G,k)$, the following result is obtained directly.

Lemma 2 *For any graph G with $xy \in E(G)$, and any integer $k \geq 1$,*

$$N(G,k) = N(G-xy,k) + N(G \bullet xy,k). \tag{10}$$

Lemma 3[3] *Let G be a graph. If $uv \in E(G)$ and uv is not in any triangularity of G, then*

$$h(G,\mu) = h(G-uv,\mu) + \mu h(G-\{u,v\},\mu).$$

Let $\mathcal{W} = \bigcup_{\substack{n\geq 1 \\ 1\leq k\leq n-1}} W(n,k)$ and $\mathcal{S} = \bigcup_{n\geq 1} S_n$.

We denote $\bar{G}(n,k)$ the graph obtained from S_{k+1} and P_{n-k} by coinciding a 2-degree vertex of S_{k+1} with a vertex of degree one in P_{n-k}.

For any $W(n,k) \in \mathcal{W}$ with $k \leq n-2$, from Lemma 2 and Figure.1, the eq.(7) in Theorem 5 holds.

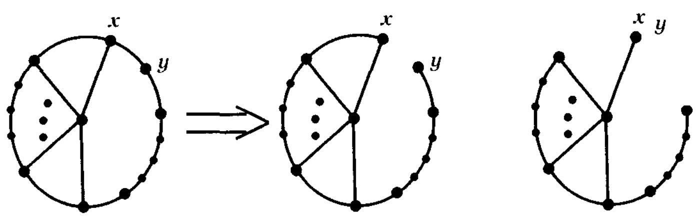

Figure 1

It can be seen from eq.(7) that the number of ideal subgraphs of $W(n,k)$ is determined by the number of ideal subgraphs of $G(n,k)$ and that of ideal subgraphs $G(n-2,k-1)$, thus, $N(G(n,k),r)$ has to be determined firstly.

Lemma 4 *The adjoint polynomial $h(G(n,k),\mu)$ of the graph $G(n,k)$ has the following recursion formula:*

$$h(G(n,k),\mu) = \mu[h(G(n-1,k),\mu) + h(G(n-2,k),\mu)] \ (1 \leq k \leq n-3).$$

The proof is shown from Lemma 3 and Figure 2.

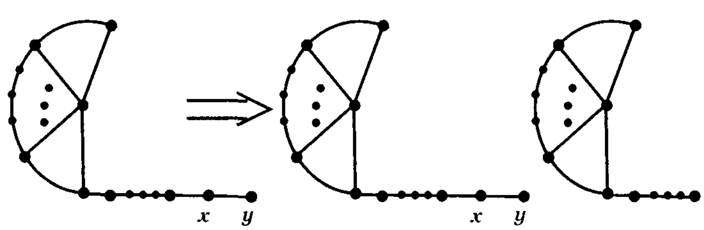

Figure 2

Lemma 5 *The adjoint polynomial $h(G(k+2,k),\mu)$ of the graph $G(k+2,k)$ is determined as follows.*

$$h(G(k+2,k),\mu) = \mu[h(S_{k+1},\mu) + h(S_k,\mu)].$$

Proof From the definition, $G(k+2,k)$ is obtained by S_{k+1} and P_2. Let two vertices of P_2 be u and v respectively. Then we have

$$G(k+2,k) - uv = G(k+1,k) \cup \{\theta\},$$

where θ is the vertex graph which has only one ideal subgraph, which implies that

$$h(G(k+2,k) - uv, \mu) = \mu h(G(k+1,k),\mu),$$

while $h(G(k+2,k) - \{u,v\},\mu) = h(G(k,k-1),\mu)$. By considering $G(i,i-1) = S_i$, the lemma is proved.

What follows is the proof of Theorem 5.

Let $\mathcal{G} = \sum_{\substack{n\geq 1\\ 1\leq k\leq n-1}} G(n,k)$ and the adjoint sum of $\mathcal{G}$ be

$$g = g_{\mathcal{G}}(x,y,\mu) = \sum_{G\in\mathcal{G}} h(G,\mu)x^{n(G)}y^{k(G)},$$

where $n(G)$ is the order of G, and $k(G)$ is the spokes of G.

Since for any fix positive integer n and k, there is only one element $G(n,k)$ in $\mathcal{G}$, thus we have

$$\begin{aligned} g &= \sum_{n\geq k+1} h(G(n,k),\mu)x^n y^k \\ &= \sum_{n\geq k+3} h(G(n,k),\mu)x^n y^k + h(G(k+2,k),\mu)x^{k+2}y^k + h(G(k+1,k),\mu)x^{k+1}y^k. \end{aligned}$$

From Lemma 4,

$$\begin{aligned} \sum_{n\geq k+3} h(G(n,k),\mu)x^n y^k &= \sum_{n\geq k+3} \mu(h(G(n-1,k),\mu) + h(G(n-2,k),\mu))x^n y^k \\ &= \mu[\text{ß}n \geq k+2h(G(n,k),\mu)x^{n+1}y^k + \sum_{n\geq k+1} h(G(n,k),\mu)]x^{n+2}y^k \\ &= \mu[xg - h(G(k+1,k),\mu)x^{k+1}y^k] + \mu x^2 g. \end{aligned}$$

After grouping, it becomes

$$[1-\mu x(1+x)]g = h(G(k+2,k),\mu)x^{k+2}y^k + (1-\mu x)h(G(k+1,k),\mu)x^{k+1}y^k, \qquad (11)$$

By applying Lemma 5 for $h(G(k+2,k),\mu)$, the following formula is derived from (11)

$$[1-\mu x(1+x)]g = h(S_{k+1},\mu)x^{k+1}y^k + \mu h(S_k,\mu)x^{k+2}y^k.$$

Since $\frac{1}{1-\mu x(1+x)} = \sum_{i\geq 0}\sum_{j=0}^{i}\mu^i\binom{i}{j}x^{i+j}$, g is the sum of two terms, the first term is

$$\sum_{i\geq 0}\sum_{j=0}^{i}\mu^i\binom{i}{j}h(S_{k+1},\mu)x^{i+j+k+1}y^k,$$

while the second term is

$$\sum_{i\geq 0}\sum_{j=0}^{i}\mu^{i+1}\binom{i}{j}h(S_k,\mu)x^{i+j+k+2}y^k.$$

Let $i+j+k+1=n$ for the first term, $i+j+k+2=n$ for the second term. Considering $\binom{l}{a}=0$ for $a<0$,

$$g=\sum_{n\geq k+1}\sum_{i\geq 0}^{n-k-1}x^ny^k\mu^i\left[\binom{i}{n-k-i-1}h(S_{k+1},\mu)+\binom{i}{n-k-i-2}\mu h(S_k,\mu)\right].$$

Thus

$$h(G(n,k),\mu)=\sum_{i\geq 0}^{n-k-1}\mu^i\left[\binom{i}{n-k-i-1}h(S_{k+1},\mu)+\binom{i}{n-k-i-2}\mu h(S_k,\mu)\right].$$

Since $h(S_n,\mu)=\sum_{r\geq 1}N(S_n,r)\mu^r$ and let $r+i=m$, we have

$$h(G(n,k),\mu)=\sum_{m\geq 1}\sum_{i\geq 0}^{m-1}\mu^m\left[\binom{i}{n-k-i-1}N(S_{k+1},m-i)+\binom{i}{n-k-i-2}N(S_k,m-i)\right.$$

From the definition of adjoint polynomial, the coefficient of μ^m in $h(G(n,k),\mu)$ is $N(G(n,k),m)$, i.e.,

$$N(G(n,k),k)=\sum_{i\geq 0}^{m-1}\left[\binom{i}{n-k-i-1}N(S_{k+1},m-i)+\binom{i-1}{n-k-i-1}N(S_k,m-i)\right],$$

which is equivalent to (8). This Theorem is proved completely.

Note Substitute $k=1$ for (8) and from Theorem 3, we have

$$\begin{aligned}N(G(n,1),m)&=\frac{(m-3)!(m-2)}{(n-m)!(2m-n-2)!}+\frac{(m-2)!(m-1)}{(n-m-1)!(2m-n)!}+\frac{(2m-n)(m-2)!}{(n-m-1)!(2m-n)!}\\&=\binom{m}{n-m}.\end{aligned}$$

Also since $G(n,1)=P_n$, this is just the conclusion of Theorem 1.

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去掉一些连续弦的轮图的补图的色多项式

郝荣霞， 刘彦佩

(北方交通大学数学系，北京 100044)

摘 要: 本文给出计算图的色多项式的新方法. 特别的，对轮图中去掉一些连续弦后所得到的图的补图，给出了它的色多项式的计算公式.

关键词: 色多项式；轮图；补图.