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On Minus Domination and Signed Domination in Graphs *

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Abstract: In this paper we obtain some lower bounds for minus and signed domination numbers. We also prove and generalize a conjecture on the minus domination number for bipartite graph of order n, which was proposed by Jean Dunbar et al [1].

Key words: minus dominating function; minus domination number; signed dominating function; signed domination number.

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1. Introduction

We use Bondy and Murty[2] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph, $V(G)$ and $E(G)$ be the set of vertices and the set of edges in G, respectively. For a vertex $v \in V(G), N_G(v) = \{u \in V(G)|uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ are the open and the closed neighborhood of v in G. For simplicity, we sometimes write $N(v)$ and $N[v]$ for $N_G(v)$ and $N_G[v]$ respeclively. For a subset $S \subseteq V(G)$, denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G)\backslash S]$. If $A, B \subseteq V(G)$ and $A \cap B = \emptyset$, then $E(A, B) = \{uv \in E(G)|u \in A$ and $v \in B\}$. If H is a subgraph of G, we write $H \subseteq G$; If H is an induced subgraph of G, we write $H \leq G$.

For a graph $G = (V, E)$, a subset $S \subseteq V$ is said to be a dominating set of G if for all $v \in V - S$, V is adjacent to some vertex of S. The domination number of G, denoted by $\gamma(G)$, is defined as the minimum cardinality of a dominating set of G.

There are many variations for the concept of domination in graphs. We are interested in the signed domination and minus domination in graphs.

Definition 1 *Given a graph $G = (V, E)$ and a subset $D \subseteq R$, a real-valued function $f : V \to D$ is said to be a D-domination function of G if $\sum_{u \in N[v]} f(u) \geq 1$ for every*

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$v \in V$. *The D-domination number of G is defined as* $\gamma_D(G) = \{\sum_{v \in V(G)} f(v) | f$ *is a D-domination function of* $G\}$.

From the above definition we can easily see the following facts:

(i) If $D_1 = \{0,1\}$, then a D_1-domination function is a dominating function and $\gamma_{D_1}(G) = \gamma(G)$ is the domination number of G.

(ii) If $D_2 = \{-1,1\}$, then a D_2-domination function is a signed dominating function and $\gamma_{D_2}(G) = \gamma_s(G)$ is the signed domination number of G.

(iii) If $D_3 = \{-1,0,1\}$, then a D_3-domination function is a minus dominating function and $\gamma_{D_3}(G) = \gamma^-(G)$ is the minus domination number of G.

For convenience, a signed (minus, resp.) dominating function f of G is called a γ_s-function (γ^--function) of G if $\sum_{v \in V(G)} f(v) = \gamma_s(G)(\gamma^-(G))$.

Obviously, a signed dominating function is also a minus dominating function. Thus, we have the following

Lemma 2 *For any graph* $G, \gamma_s(G) \geq \gamma^-(G)$.

In [3] we obtained some lower bounds for signed domination number of graphs, one of which was stated as follows:

Lemma 3[3] *For any graph G of order n, we have*

$$\gamma_s(G) \geq 2\lceil \frac{-1+\sqrt{1+8n}}{2} \rceil - n$$

and this bound is sharp.

For the minus domination number of a graph, J.Dunbar et al.[1] obtained some results for several classes of graphs. They asked if there exists a graph G with girth m and $\gamma^-(G) \leq k$ for every negative integer k and positive integer m ? J.Lee et al.[4] gave a positive answer to this problem. In addition, a conjecture was posed in [1] as follows:

Conjecture 4[1] *If G is a bipartite graph of order n, then* $\gamma^-(G) \geq 4(\sqrt{n+1}-1)-n$.

In this note, we prove and generalize this conjecture, and obtain some new lower bounds for minus (signed) domination numbers of graphs. In addition, we also give a method to find the lower bounds of γ_s(G) and $\gamma^-(G)$ for all graphs G.

2. Main results

We first give a method to find the lower bound of $\gamma_s(G)$ for every graph G.

Theorem 5 *For any graph G of order n, let*

$$\varphi_s(G) = \max\{|E(H)| | H \subseteq G \text{ and } |V(H)| = s\} \text{ and } (G) = \min\{s | s + \varphi_s(G) \geq n\}.$$

Then $\gamma_s(G) \geq 2S(G) - n$.

Proof Let f be a γ_s-function of G. Define

$$A = \{v \in V(G) | f(v) = 1\}, \quad B = \{v \in V(G) | f(v) = -1\}.$$

Obviously, $V(G) = A \cup B$ and $A \cap B = \emptyset$. Let $|A| = s$. Then $|B| = n - s$ and then $\gamma_s(G) = |A| - |B| = 2s - n$.

By the definition of a signed dominating function, it is clear that $|N_G(v) \cap A \geq 2$ for each $v \in B$. So, we have

$$|E(A,B)| \geq 2|B| = 2(n-s). \tag{1}$$

Let $G_1 = G[A]$, and $d_{G_1}(v)$ be the degree of v in G_1 (if $v \in V(G_1)$). For each $v \in A$, v is adjacent to at most $d_{G_1}(v)$ vertices of B. That is, $|N_G(v) \cap B| \leq d_{G_1}(v)$ for every $v \in A$.

From the definition of $\varphi_s(G)$, we see that $|E(G_1)| \leq \varphi_s(G)$. Thus we have

$$|E(A,B)| \leq \sum_{v \in A} d_{G_1}(v) = 2|E(G_1)| \leq 2\varphi_s(G).$$

Together with (1), we have

$$s + \varphi_s(G) \geq n. \tag{2}$$

Since $S(G)$ is defined as the minimum integer s satisfying (2). Thus $s \geq S(G)$. We have $\gamma_s(G) = 2s - n \geq 2S(G) - n$. The proof is complete.

Remark For any graph G of order n, since $\varphi_s(G) \leq \binom{s}{2}$ holds for every integer $s(1 \leq s \leq n)$, if $s + \varphi_s(G) \geq n$, then $s + \binom{s}{2} \geq n$, namely, $s \geq \frac{-1+\sqrt{1+8n}}{2}$, which implies $S(G) \geq \frac{-1+\sqrt{1+8n}}{2}$. Note that $S(G)$ is an integer, we see the result of Lemma 3.

Using Theorem 5, we can easily obtain the lower bounds of $\gamma_s(G)$ for some special graphs G, such as trees, planar graphs, triangle-free graphs, etc.

A graph G is called triangle-free if $K_3 \not\subset G$.

Lemma 6[5] *For any triangle-free graph G of order n,$|E(G)| \leq \frac{n^2}{4}$.*

Lemma 7[5] *For any planar graph G of order $n(n \geq 3)$, $|E(G)| \leq 3n - 6$.*

Corollary 8 *For any triangle-free graph G of order n, then $\gamma_s(G) \geq 2\lceil 2(\sqrt{n+1}-1)\rceil - n$ and this bound is sharp.*

Proof Since any subgraph of G is also triangle-free, by Lemma 6, we have $\varphi_s(G) \leq \frac{s^2}{4}$.

For any positive integer s satisfying that $s + \varphi_s(G) \geq n$, then $s + \frac{s^2}{4} \geq n$, namely. $s \geq 2(\sqrt{n+1}-1)$. By the definition of $S(G)$ in Theorem 5, and note that $S(G)$ is an integer, we have $S(G) \geq \lceil 2(\sqrt{n+1}-1)\rceil$. By Theorem 5, $\gamma_s(G) \geq 2S(G) - n \geq 2\lceil 2(\sqrt{n+1}-1)\rceil - n$. And this bound is sharp (see [1], page 46), we have completed the proof of Corollary 8.

Corollary 9 *For any planar graph G of order $n(n \geq 4)$, then $\gamma_s(G) \geq 2\lceil \frac{n+6}{4} \rceil - n$.*

Proof By Lemma 7, for any integer $s \geq 1$, we have

$$\varphi_s(G) \leq \begin{cases} s-1, & \text{when } s = 1 \text{ or } 2; \\ 3s-6, & \text{when } s \geq 3. \end{cases}$$

If $s + \varphi_s(G) \geq n$, note that $n \geq 4$ and then $s \geq 3$, we have $s + 3s - 6 \geq n$, namely, $s \geq \lceil \frac{n+6}{4} \rceil$.

From the definition of $S(G)$ in Theorem 5, we see that $S(G) \geq \lceil \frac{n+6}{4} \rceil$, by Theorem 5, we have completed the proof of Corollary 9.

Next we consider the lower bounds of minus domination numbers in graphs. ∎

Theorem 10 *For any graph G of order n, Let $\pi(G) = \min\{\gamma_s(H) | H \leq G\}$, then $\gamma^-(G) \geq \pi(G)$.*

Proof Let f be a γ^--function of G, and then $\gamma^-(G) = \sum_{v \in V(G)} f(v)$. Let

$$X_0 = \{v \in V(G) | f(v) = 0\} \text{ and } G_1 = G - X_0.$$

It is easy to see that $f_1 = f|_{G_1}$ is a signed dominating function of G_1. Thus we have $\gamma^-(G) = \sum_{v \in V(G)} f(v) = \sum_{v \in V(G_1)} f_1(v) \geq \min\{\gamma_s(H) | H \leq G\} = \pi(G)$. This proof is complete.

The above theorem give a method to find the lower bound of $\gamma^-(G)$.

Corollary 11 *For any graph G of order $n (n \geq 3), \gamma^-(G) \geq \lceil \sqrt{1+8n} - 1 \rceil - n$.*

Proof By Lemma 3, for any induced subgraph H of G, let $|V(H)| = s (1 \leq s \leq n), \gamma_s(H) \geq 2\lceil \frac{-1+\sqrt{1+8s}}{2} \rceil - s \geq \sqrt{1+8s} - 1 - s$.

By Theorem 10, and note that $n \geq 3$, then $\gamma^-(G) \geq \min\{\gamma_s(H) | H \leq G\} \geq \min\{\sqrt{1+8s} - 1 - s | 1 \leq s \leq n\} = \sqrt{1+8s} - 1 - n$. Hence, we have

$$\gamma^-(G) \geq \lceil \sqrt{1+8n} - 1 \rceil - n,$$

this proof is complete.

Corollary 12 *For any triangle-free graph G of order n, then*

$$\gamma^-(G) \geq 4(\sqrt{n+1} - 1) - n$$

and this bound is sharp.

Proof By Theorem 10 and Corollary 8, we have

$$\begin{aligned}\gamma^-(G) \geq \pi(G) = \min\{\gamma_s(H) | H \leq G\} &\geq \min\{2\lceil 2(\sqrt{s+1} - 1) \rceil - s | 1 \leq s \leq n\} \\ &\geq \min\{4(\sqrt{s+1} - 1) - s | 1 \leq s \leq n\}\end{aligned}$$

Note that $\gamma^-(G)$ is an integer, thus we have

$$\gamma^-(G) \geq \min\{4(\sqrt{s+1} - 1) - s | 1 \leq s \leq n\} = \lceil 4(\sqrt{n+1} - 1) - n \rceil \geq 4(\sqrt{n+1} - 1) - n$$

and this bound is sharp (see [1]). We have completed the proof of Corollary 12.

From Corollary 12 we see that $\gamma^-(G) \geq 4(\sqrt{n+1} - 1) - n$ holds for all bipartite graphs G of order n. Thus we have proved and generalized Conjecture 4. ∎

Corollary 13 *For any planar graph G of order $n (n \geq 4)$, then $\gamma^-(G) \geq \lceil \frac{n+6}{2} \rceil - n$.*

Proof By Theorem 10, we have

$$\gamma^-(G) \geq \pi(G) = \min\{\gamma_s(H) | H \leq G\}.$$

Choose such an induced subgraph $H_0 \leq G$ that $\gamma_s(H_0) = \min\{\gamma_s(H)|H \leq G\}$.

If $1 \leq |V(H_0)| \leq 3$; Obviously, $\gamma^-(G) \geq \gamma_s(H_0) \geq 1 \geq \lceil\frac{n+6}{2}\rceil - n$ (note that $n \geq 4$).

If $|V(H_0)| \geq 4$; Since H_0 is a planar graph with at least four vertices, by Corollary 9, we have

$$\gamma^-(G) \geq \gamma_s(H_0) \geq 2\lceil\frac{n+6}{2}\rceil - n \geq \frac{n+6}{2} - n.$$

Note that $\gamma^-(G)$ is an integer, we have completed the proof of Corollary 13.

Finally, we end this note with the following

Problem Determine $B(n, g) = \min\{\gamma^-(G)|G$ is a graph with n vertices and girth $g(G) \geq g\}$ for all integers n and $g(3 \leq g \leq n)$.

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关于图的减控制与符号控制

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摘 要: 给定一个图 $G=(V,E)$, 一个函数 $f: V \to \{-1,0,1\}$ 被称为 G 的减控制函数, 如果对任意 $v \in V(G)$ 均有 $\sum_{u \in N[v]} f(u) \geq 1$. G 的减控制数定义为 $\gamma^-(G) = \min\{\sum_{v \in V} f(v)|f$ 是 G 的减控制函数 $\}$. 图 G 的符号控制函数的正如减控制函数, 差别是 $\{-1,0,1\}$ 换成 $\{-1,1\}$. 符号控制数 $\gamma_s(G)$ 是类似的. 本文获得了 $\gamma^- G)$ 和 $\gamma_s(G)$ 的一些下界. 同时也证明并推广了 Jean Dunbar[1] 等提出的一个猜想, 即对任意 n 阶 2 部图 G, 均有 $\gamma^-(G) \geq 4(\sqrt{n+1}-1)-n$ 成立.

关键词: 减控制函数; 减控制数; 符号控制函数; 符号控制数.