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On the Aleksandrov Problem of Isometric Mapping *

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Abstract: This paper deals with some problems of isometry to extend Aleksandrov problem, Benz's result and Mazur-Ulam theorem.

Key words: isometry; k-isometry; (F)-space; DOPP; SDOPP.

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1. Introduction

The beginning of the study of linear extension of isometric mapping is due to Mazur-Ulam's famous theorem gotten in 1932. The main goal of the study is to answer the following question:

If X, Y are two real normed linear spaces, $U : X \to Y$ is an isometry, is the U an affine?

For more than 70 years, many mathematicians have studied this problem in different aspects.

Mazur-Ulam's theorem[1] gave the question a positive answer for surjective mapping, i.e. if X, Y are two real normed linear spaces and $f : X \to Y$ is an surjective isometry, then f is affine.

In 1968, T.Figiel[2] considered the "into mapping", and gave the following famous result that took the condition "onto" out: Let X, Y be two real Banach spaces. If $F : X \to Y$ is a injective isometric mapping and $F(0) = 0$, then there exists a continuous linear mapping $f : \overline{\mathrm{span} F(X)} \to X$ such that $f \circ F$ is an identify and $\|f|_{\mathrm{span}\{F(X)\}}\| = 1$.

In 1971, Baker[3] proved that if Y is strictly convex, the answer is also positive even if the condition "onto" was taken out .

In 1982, Rolewicz[4] extended Mazur-Ulam's theorem to quasi-normed space (F-space).

Another correlative problem is "one distance preserving mapping" (Aleksandrove problem). Suppose $(X, d), (Y, d)$ are two metric linear spaces. A mapping $f : X \to Y$ is said to

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have one distance preserving property (DOPP), if for any $x, y \in X$, the equality $d(x,y)=1$ implies the equality $d(f(x), f(y))=1$.

Mapping $f: X \to Y$ is said to have strong one distance preserving property(SDOPP), if for any $x, y \in X$, $d(x,y)=1$ if and only if $d(f(x), f(y))=1$.

In 1970[5], Aleksandrove proposed the following problem: If one distance preserving mapping is an isometry?

In 1993, Benz[7], Mielnik and Rassias[3] got some results. We have obtain some results on this problem[8]. In the next two sections of this paper,we will do further research on Aleksandrove'problem and Rolewicz'theorem.

2. Aleksandrove problem

Lemma 2.1[5] *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Suppose $f: X \to Y$ is a surjective mapping which satisfies (SDOPP). Then, f preserves distance n in both directions for any positive integer n.*

Theorem 2.2[5] *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Suppose $f: X \to Y$ is a Lipschitz mapping with $K=1$, i.e. $\|f(x)-f(y)\| \le \|x-y\|$ for all $x, y \in X$. Assume f is also a surjective mapping satisfying (SDOPP). Then f is an isometry. Thus f is a linear isometry to translation.*

Remark 2.3 In the above theorem, the condition of $\|f(x)-f(y)\| \le \|x-y\|$ for all $x, y \in X$ can be substituted by $\|f(x)-f(y)\| \le \|x-y\|$ for $x, y \in X$ with $\|x-y\| \le 1$.

Proof Assume $\|x-y\| > 1$. Then there exists $n_0 \in \mathbf{N}$ such that $n_0 \le \|x-y\| \le n_0+1$, so

$$\|x-y\| - n_0 \le 1.$$

Let

$$z = x + \frac{n_0}{\|x-y\|}(y-x),$$

then $\|z-x\| = n_0$ and

$$\|z-y\| = \|(x-y) + \frac{n_0}{\|y-x\|}(y-x)\| = \|x-y\| - n_0 \le 1.$$

By the Lemma and the condition we know $\|f(z)-f(x)\| = n_0$ and $\|f(z)-f(y)\| \le \|z-y\|$, thus

$$\begin{aligned}\|f(x)-f(y)\| &\le \|f(x)-f(z)\| + \|f(z)-f(y)\| \\ &\le n_0 + \|z-y\| = n_0 + \|x-y\| - n_0 = \|x-y\|.\end{aligned}$$

Here we use the original condition that f satisfies (DOPP), more, preserving n for any positive integer, but we can easily find that the condition is important only for the proof that f is an isometry. Moreover, we have following Theorem:

Theorem 2.4 *Let X,Y be two normed spaces and $f: X \to Y$ satisfy (DOPP). If $\|f(x)-f(y)\| \le \|x-y\|$ for $x, y \in X$ with $\|x-y\| \le 1$, then*

$$\|f(x)-f(y)\| \le \|x-y\| \text{ for all } x, y \in X,$$

and especially

$$\|f(x)-f(y)\|=\|x-y\|,\ \textit{for all}\ \|x-y\|\le 1.$$

Proof Firstly, we prove that if $\|x-y\|\le 1$ then $\|f(x)-f(y)\|=\|x-y\|$. Assume

$$\|f(x)-f(y)\|<\|x-y\|$$

and let

$$z=x+\frac{1}{\|y-x\|}(y-x).$$

It is clear that

$$\|z-x\|=1\ \text{ and }\ \|z-y\|=1-\|x-y\|\le 1.$$

Then

$$\|f(z)-f(x)\|=1\ \text{ and }\ \|f(z)-f(y)\|\le 1-\|x-y\|.$$

On the other hand, we have

$$\|f(z)-f(x)\|\le\|f(z)-f(y)\|+\|f(y)-f(x)\|<1-\|x-y\|+\|x-y\|=1.$$

This contradicts the equality $\|f(z)-f(x)\|=1$, hence $\|f(x)-f(y)\|=\|x-y\|$.

Secondly, we prove

$$\|f(x)-f(y)\|\le\|x-y\|,\quad \forall x,y\in X.$$

For $x,y\in X$, we can find two positive integers m,n with $\|x-y\|\le\frac{m}{n}$. If $m=1$, the result is obvious, so we suppose $m\ge 2$. We can find a finite sequence of vectors $x=z_0,z_1,\cdots,z_m=y$ such that $\|z_i-z_{i+1}\|\le\frac{1}{n}$. It follows that

$$\|f(x)-f(y)\|\le\sum_{i=0}^{m-1}\|f(z_i)-f(z_{i+1})\|\le\sum_{i=0}^{m-1}\|z_i-z_{i+1}\|=\frac{m}{n}.$$

Thus

$$\|f(x)-f(y)\|\le\frac{m}{n},$$

and

$$\|f(x)-f(y)\|\le\|x-y\|,\quad \forall x,y\in X.$$

Walter Benz Theorem[7] *Let X,Y be two real normed linear space,* $\dim X\ge 2$ *and Y be a strictly convex. If there exist $\rho\in R$ with $\rho>0$, and $n\in N$ with $n>1$ and there exists a mapping $f:X\to Y$ satisfying the following conditions:*

(1) for any $a\in X$ with $\|a\|<1$, there exists some $b\in X$ such that $\|a-b\|=1=\|a+b\|$;

(2) $\|x-y\|=\rho\Rightarrow\|f(x)-f(y)\|\le\rho$;

(3) $\|x-y\|=n\rho\Rightarrow\|f(x)-f(y)\|\ge n\rho$,

then f is an affine mapping.

Now we give next theorem:

Theorem 2.5 *Suppose X, Y are two real normed linear spaces and Y is strictly convex. If there exists $n \in N, n > 1$ and a mapping $f: X \to Y$ such that for any $x, y \in X$ we have*

(1) $\|x-y\| \leq 1 \Rightarrow \|f(x)-f(y)\| \leq \|x-y\|$;

(2) $\|x-y\| = n \Rightarrow \|f(x)-f(y)\| \geq n$,

then f is an affine isometry.

We need the following lemma:

Lemma 2.6[8] *Let X, Y be two real normed linear spaces and Y is strictly convex. If*

(1) $f: X \to Y$ satisfies (DOPP);

(2) $\|x-y\| \leq 1 \Rightarrow \|f(x)-f(y)\| \leq \|x-y\|$,

then f is an affine isometry.

Proof of Theorem 2.5 We need only to prove f satisfies (DOPP). Let $\|x-y\| = 1$ and $p_i = y + i(x-y), i = 0, 1, 2, \cdots, n$, $p_0 = y, p_1 = x$. Then $\|p_n - y\| = n$ and $\|p_i - p_{i-1}\| = 1$, $i = 0, 1, 2, \cdots, n$, so by the conditions of the theorem we have $\|f(p_n) - f(y)\| \geq n$ and $\|f(p_i) - f(p_{i-1})\| \leq 1$ for $i = 0, 1, 2, \cdots, n$, thus

$$n \leq \|f(p_n) - f(y)\| \leq \sum_{i=1}^{n} \|f(p_{n+1-i}) - f(p_{n-i})\| \leq n.$$

So

$$\sum_{i=1}^{n} \|f(p_{n+1-i}) - f(p_{n-i})\| = n,$$

and hence

$$\|f(p_{n+1-i}) - f(p_{n-i})\| = 1.$$

This implies $\|f(x) - f(y)\| = 1$, or f satisfies (DOPP).

According to the proof given by T.M.Rassias we have next result [3]:

Corollary 2.7[5] *Let X, Y be two real normed linear spaces, $\dim X > 1$ and Y is strictly convex. Suppose $f: X \to Y$ preserve n for any positive integer $n \in \mathbf{N}$. Then*

$$\|f(x) - f(y)\| \leq \|x - y\|, \forall x, y \in X.$$

Moreover we obtain next result:

Corollary 2.8 *Let X, Y be two real normed linear spaces, Y is strictly convex and the mapping $f: X \to Y$ satisfies thes following conditions:*

(1) f satisfies (DOPP);

(2) $\|x-y\| \leq \|f(x)-f(y)\|$ for any $x, y \in X$,

then f is an isometry.

Proof For any $n \in \mathbf{N}$, assume $\|x - y\| = n$, we can find a finite sequence of vectors $x = z_0, z_1, ..., z_n = y$ such that $\|z_i - z_{i+i}\| = 1$. It follows that

$$\|f(x) - f(y)\| \leq \Sigma_{i=0}^{n-1} \|f(z_i) - f(z_{i+1})\| = n.$$

Thus $n = \|x-y\| \le \|f(x)-f(y)\| \le n$, so f preserve n for any positive integer. Moreover, by the above Corollary, we have

$$\|f(x) - f(y)\| \le \|x - y\| \text{ for any } x, y \in X,$$

thus

$$\|f(x) - f(y)\| = \|x - y\| \text{ for any } x, y \in X.$$

3. Rolewicz problem

Next, one of Rolewicz Theorems[4] will be extended. We use Mazur-Ulam method to obtain the main result: Let X and Y be two real locally bounded linear spaces, and $U : X \to Y$ be a surjective map satisfying $\|k(U(x)-U(y))\| = \|k(x-y)\|$ $(k = \frac{1}{2^n}, n \in \{0\} \cup N)$. Then U is an affine mapping from X to Y.

A function $\|.\|$ on a linear space X is called a F-norm if

(1) $\|x\| = 0$ if and only if $x = 0$;
(2) $\|ax\| = \|x\|$ for all a with $|a| = 1$;
(3) $\|x + y\| \le \|x\| + \|y\|$;
(4) $\|a_n x\| \to 0$ provided $a_n \to 0$;
(5) $\|ax_n\| \to 0$ provided $x_n \to 0$.

A linear space equipped with an F-norm $\|.\|$ is called an F^*-space (see Banach, 1932.) An F^*-space X is said to be locally bounded if it has a bounded neighborhood of zero. It is well known that X is locally bounded if and only if X has an equivalent β-norm[2].

Let X and Y be two real F^*-spaces. $U : X \to Y$ is called k- isometry if $\|k(Ux - Uy)\| = \|k(x-y)\|$ for all $x, y \in X$.

Let X and Y be two real normed linear spaces. If $U : X \to Y$ is an isometry, is U an affine? Mazur and Ulam[1] proved that as U is a surjection, it is necessarily an affine. Baker[5] proved that as Y is a strictly convex space, U is necessarily an affine. Rolewicz obtained the following theorem[2]:

Theorem 3.1 *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be two real locally bounded spaces. If $U : X \to Y$ is a surjective map satisfying $\|k(U(x) - U(y))\| = \|k(x-y)\|$ for all $x, y \in X$ and $k \in R^+$, then U is an affine.*

Now we use Mazur-Ulam's method to give an extended proposition.

Theorem 3.2 *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be two real locally bounded spaces. If $U : X \to Y$ is a surjective map satisfying $\|k(U(x) - U(y))\| = \|k(x-y)\|$ for all $x, y \in X$ and $k = \frac{1}{2^n}, (n \in \{0\} \cup N)$, then U is an affine.*

Proof Let x_1 and $x_2 \in (X, \|.\|)$ and

$$H_1 = \{x \in X \mid \ \|x - x_1\| = \|x - x_2\| = \|\frac{x_1 - x_2}{2}\|\},$$

and for each $n > 1$, define

$$H_n = \{x \in H_{n-1} \mid \|\frac{x-z}{2^{k-1}}\| \le d(\frac{H^{n-1}}{2^k}), \ \forall k \in N, z \in H_{n-1}\},$$

here $d(A)=\sup\{\|x-y\| \mid \forall x,y\in A\}$ denotes the diameter of A. We show $\lim_{n\to\infty} d(H_n)=0$ now. Assume $H_n\neq\emptyset$ for all $n\in N$.

For any $x_1,x_2\in H_n\subset H_{n-1}$, we have $\|\frac{x_1-x_2}{2^{k-1}}\|\le d(\frac{H_{n-1}}{2^k})(k\in N)$, it follows that $d(\frac{H_n}{2^{k-1}})\le d(\frac{H_{n-1}}{2^k})(k\in N)$, thus

$$d(H_n)\le d(\frac{H_{n-1}}{2})\le\ldots\le d(\frac{H_1}{2^{n-1}}).$$

For $x,y\in H_1$, it follows that $\|x-y\|\le\|x-x_1\|+\|y-x_1\|=2\|\frac{x_1-x_2}{2}\|$ and $d(H_1)\le 2\|\frac{x_1-x_2}{2}\|$.

Since X is locally bounded and has an equivalent β-norm, it follows that

$$\lim_{n\to\infty} d(\frac{H_1}{2^{n-1}})=0 \quad\text{and}\quad \lim_{n\to\infty} d(H_n)=0.$$

Next we show $\bigcap_{n=1}^{\infty}H_n\neq\emptyset$, thus $\bigcap H_n$ contains a unique element.

1. For any $x\in X$, let $\bar{x}=x_1+x_2-x$. Suppose $x\in H_n$, then $\bar{x}\in H_n$, which follows by induction on n. In fact, when $n=1$, if $x\in H_1$, then

$$\|\bar{x}-x_1\|=\|\bar{x}-x_2\|=\|\frac{x_1-x_2}{2}\|,$$

this implies $\bar{x}\in H_1$. Assume $n>1$ and the conclusion is testified for $n-1$. Let for any $x\in H_n$, then for any $z\in H_{n-1}$, by the inductive hypothesis we have $\bar{z}\in H_{n-1}$, and by the definition of H_n we have

$$\|\frac{\bar{x}-z}{2^{k-1}}\|=\|\frac{x_1+x_2-x-z}{2^{k-1}}\|=\|\frac{x-\bar{z}}{2^{k-1}}\|\le d(\frac{H_{n-1}}{2^k}),\quad \forall k\in N.$$

It follows that $\bar{x}\in H_n$. This completes the inductive step.

2. Our next goal is to show $\xi=\frac{1}{2}(x_1+x_2)\in\bigcap H_n$ by induction. Since $\xi\in H_1$, assume $\xi\in H_{n-1}$. Then for any $x\in H_{n-1}$, from above step we have $\bar{x}\in H_{n-1}$ and

$$\|\frac{\xi-x}{2^{k-1}}\|=\|\frac{x_1+x_2-x-x}{2^k}\|\le d(\frac{H_{n-1}}{2^k}),\quad \forall k\in N.$$

It follows that $\xi\in H_n$. This completes the induction, and we have shown $\xi\in\bigcap H_n$. We called ξ the center of x_1 and x_2. Similarly $\frac{1}{2}(Ux_1+Ux_2)$ is the center of Ux_1 and Ux_2.

Now we show $U(\frac{x_1+x_2}{2})$ is the center of Ux_1 and Ux_2, and then by the uniqueness we know $\frac{1}{2}(Ux_1+Ux_2)=U(\frac{x_1+x_2}{2})$. In fact, let $\bar{H}_n$ be a subset in Y being similar to H_n, then

$$\bar{H}_1=\{y\in Y \mid \|y-Ux_1\|=\|y-Ux_2\|=\|\frac{Ux_1-Ux_2}{2}\|\},$$

$$\bar{H}_n=\{y\in\bar{H}_{n-1} \mid \|\frac{y-z}{2^{k-1}}\|\le d(\frac{\bar{H}_{n-1}}{2^k}),\forall z\in\bar{H}_{n-1},k\in N\}.$$

We want to show $U(H_n)=\bar{H}_n$ by induction.

When $n=1$, for any $y\in\bar{H}_1$, since U is a surjection and satisfies

$$\|k(Ux-Uy)\|=\|k(x-y)\|\ \ (k=\frac{1}{2^n},n\in\{0\}\cup N),$$

there exists $x \in X$ such that $U(x) = y$ and

$$\|y - Ux_1\| = \|y - Ux_2\| = \|\frac{Ux_1 - Ux_2}{2}\| = \|\frac{1}{2}(x_1 - x_2)\|.$$

Thus $\|x - x_1\| = \|x - x_2\| = \|\frac{x_1 - x_2}{2}\|$ and hence $x \in H_1$. We have gotten $\bar{H}_1 \subseteq U(H_1)$. Conversely, for any $x \in H_1$, we have

$$\|Ux - Ux_1\| = \|x - x_1\| = \|\frac{x_1 - x_2}{2}\| = \|\frac{Ux_1 - Ux_2}{2}\|.$$

Thus by $\|Ux - Ux_2\| = \|\frac{Ux_1 - Ux_2}{2}\|$, we have $Ux \in \bar{H}_1$ and $\bar{H}_1 = U(H_1)$. From the fact that U is $\frac{1}{2^n}$-isometry ($n \in \{0\} \cup N$), we have $d(\frac{\bar{H}_1}{2^k}) = d(\frac{H_1}{2^k})$ ($k \in \{0\} \cup N$).

Assume $\bar{H}_{n-1} = U(H_{n-1})$. Then it is obvious that $d(\frac{\bar{H}_{n-1}}{2^k}) = d(\frac{H_{n-1}}{2^k}) \forall k \in \{0\} \cup N$. Let $y \in \bar{H}_n$, then for any $\tilde{z} \in \bar{H}_{n-1}$, we have

$$\|\frac{y - \tilde{z}}{2^{k-1}}\| \le d(\frac{\bar{H}_{n-1}}{2^k}) = d(\frac{H_{n-1}}{2^k}) \forall k \in N.$$

By the inductive hypothesis and $y \in \bar{H}_n \subset \bar{H}_{n-1}$, there exists $x \in H_{n-1}$ such that $Ux = y$. For any $z \in H_{n-1}$, we have $Uz \in \bar{H}_{n-1}$ and

$$\|\frac{x - z}{2^{k-1}}\| = \|\frac{Ux - Uz}{2^{k-1}}\| = \|\frac{y - Uz}{2^{k-1}}\| \le d(\frac{H_{n-1}}{2^k}).$$

This implies $x \in H_n$ and $\bar{H}_n \subset U(H_n)$. Conversely, for $y \in U(H_n)$, there exists $x \in H_n$ such that $y = Ux$. From $x \in H_n \subset H_{n-1}$ and the inductive hypothesis we know $y \in \bar{H}_{n-1}$. For any $\tilde{z} \in \bar{H}_{n-1}$, there exists $z \in H_{n-1}$ such that $Uz = \tilde{z}$ and

$$\|\frac{y - \tilde{z}}{2^{k-1}}\| = \|\frac{x - z}{2^{k-1}}\| \le d(\frac{H_{n-1}}{2^k}) = d(\frac{\bar{H}_{n-1}}{2^k}) \ \forall k \in N.$$

Thus $y \in \bar{H}_n$ and $\bar{H}_n = U(H_n)$. This is to say that $U(H_n) = \bar{H}_n$ for all $n \in N$.

Because $\frac{x_1 + x_2}{2} \in H_n$, we have $U(\frac{x_1 + x_2}{2}) \in \bar{H}_n$ for each $n \in N$. Thus we have that $U(\frac{x_1 + x_2}{2}) = \frac{1}{2}(Ux_1 + Ux_2)$, and furthermore U is an affine. This completes the proof of the theorem.

Obviously, we have the following corollary:

Corollary 3.3 *If X and Y are same as above, and $U : X \to Y$ is a surjective isometry with $U(\frac{1}{2}x) = \frac{1}{2}Ux$ for any $x \in X$, then U is a linear map.*

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关于等距映射的 Aleksandrov 问题

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摘 要: 本文推广了保 1 映射的 Aleksandrov 问题, 同时推广了 Benz 定理和 Mazur-Ulam 定理.

关键词: 等距; *k*- 等距; (F)- 空间; DOPP; SDOPP.