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On a Question of Matveev *

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Abstract: In this paper, we give an example of a Hausdorff star-Lindelöf space X having property (a) such that $e(X) = 2^c$. This is a partial answer to a question of Matveev.

Key words: Hausdorff star-Lindeöf space; extent.

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1. Introduction

By a space, we mean a topological space. The purpose of this paper to give a Hasudorff star-Lindelöf space X having property (a) such that $e(X) = 2^c$. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $B \subset X$, let $St(B,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap B \neq \emptyset\}$. As usual, we write $St(x,\mathcal{U})$ for $St(\{x\},\mathcal{U})$.

Definition 1[1] *A space X is star-Lindelöf if for every open cover $\mathcal{U}$ of X, there exists a countable subset $B \subset X$ such that $St(B,\mathcal{U}) = X$.*

Clearly, all Lindeöf spaces and all separable spaces are Lindelöf.

Definition 2[2] *A space X has the property (a) if for every open cover $\mathcal{U}$ of X and every dense D of X, there exists $F \subset D$ such that F is discrete closed in X and $St(F,\mathcal{U}) = X$.*

Moreover, the extent $e(X)$ of a space X is the smallest infinite cardinal κ such that every discrete closed subset of X has cardinality at most κ. The cardinality of a set A is denoted by $|A|$. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. Let ω denote the first infinite cardinal and c the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols that we do not define will be used as in [3].

2. An example on star-Lindelöf spaces

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In [4], Song gave an example that the extent of a Tychonoff star-Lindelöf space having the property (a) with $e(X) = c$ which answered Question 4.5 from [5]. Further, Matveev asked if there exists of a star-Lindelöf space X having the property (a) such that $e(X) \geq c$. The purpose of this section is to answer the former questions positively in the sphere of Hausdorff.

Next, we give a machine that produces star-Lindelöf space having the property (a). In the following, we state the definition of the machine. For a separable space X and its a countable dense subset D, we define

$$S(X,D) = X \cup (D \times \kappa^+), \quad \text{where} \quad \kappa = |X|,$$

and topogize $S(X,D)$ as follows: A basic neighborhood of $x \in X$ in $S(X,D)$ is a set of the form

$$G_{U,\alpha}(x) = \{x\} \cup ((U \cap D) \times \{\beta : \alpha < \beta < \kappa^+\})$$

for a neighborhood U of x in X and for $\alpha < \kappa^+$, and a basic neighborhood of $\langle x, \alpha \rangle \in D \times \kappa^+$ in $S(X,D)$ is a set of the form

$$G_V(\langle x, \alpha \rangle) = \{x\} \times V$$

for a neighborhood V of α in κ^+. When it is not necessary to specify D, we simply write $S(X)$ instead of $S(X,D)$.

Recall from [6] that a space X is absolutely countably compact (=acc) if for every open cover $\mathcal{U}$ of X and every dense D of X, there exists a finite subset $F \subset D$ such that $St(F,\mathcal{U}) = X$. It is known that every T_2 absolutely countably compact space is countably compact space having the propery (a) (see [2,6]). Moreover, Vaughan[7] proved that every countably compact GO-space is absolutely countably compact. Thus, every cardinal with uncountable cofinality is absolutely countably compact. By a Tychonoff space, we mean a completely regular T_1-space.

Theorem 2.1 *Let X be a separable space with a countable dense set D. Then, $S(X,D)$ is star-Lindelöf space having the property (a).*

(1) If X is a T_2-space, so is $S(X,D)$;

(2) If X is a regular space, so is $S(X,D)$;

(3) If X is a normal space, so is $S(X,D)$.

Proof Put $S = S(X,D)$. Now, we show that S is a star-Lindelöf space having the property (a) simultaneously. For this end, let $\mathcal{U}$ be an open cover of S. Let T be the set of all isolated point of κ^+ and let $F = D \times T$. Then, F is dense in S and every dense subspace of S includes F. Thus, it suffices to show that there exists a countable subset $B \subset F$ such that B is discrete closed in S and $St(B,\mathcal{U}) = S$. For each $d \in D, \{d\} \times \kappa^+$ is acc, there exists a finite subset $B_d \subset \{d\} \times T$ such that $\{d\} \times \kappa^+ \subset St(B_d,\mathcal{U})$. Let $B' = \cup\{B_d : d \in D\}$. Then, $D \times \kappa^+ \subset St(B',\mathcal{U})$. For each $x \in X$, take $U_x \in \mathcal{U}$ with $x \in U_x$ and fix $\alpha_x < \kappa^+$ and $d_x \in D$ such that $\{\langle d_x, \alpha \rangle : \alpha_x < \alpha < \kappa^+\} \subset U$. For each $d \in D$, let $X_d = \{x \in X : d_x = d\}$ and choose $\beta_d \in T$ with $\beta_d > \sup\{\alpha_x : x \in X_d\}$. Then, $X_d \subset St(\langle d, \beta_d \rangle,\mathcal{U})$. Thus, if we put $B'' = \{\langle d, \beta_d \rangle : d \in D\}$, then $X \in St(B'',\mathcal{U})$.

Let $B = B' \cup B''$. Then B is a countable subset of F such that $S = St(B,\mathcal{U})$. Since $B \cap (\{d\} \times \kappa^+)$ is finite for each $d \in D$, B is discrete and closed in S, which proves that S is a star-Lindelöf space having the property (a).

The proof of the statements (1) and (2) is obvious. We omit it here.

Finally, to prove the statement (3), assume that X is normal. Let A_0 and A_1 be disjoint closed subsets of $S(X,D)$. Since X is normal and $\kappa^+ > |X|$, we can find disjoint open subsets U_0, U_1 of X and $\alpha < \kappa^+$ such that $A_i \cap X \subset U_i$ and

$$(U_i \cup ((U_i \cap D) \times (\alpha, \kappa^+)) \cap A_{1-i} = \emptyset$$

for each $i = 0,1$. Let $X_0 = D \times \kappa^+$ and put

$$B_i = ((U_i \cap D) \times (\alpha, \kappa^+)) \cup (A_i \cap X_0) \ \text{ for } \ i = 0,1.$$

Then, B_0 and B_1 are disjoint closed in X_0. Since X_0 is normal, there exist disjoint open sets V_0 and V_1 in X_0 such that $B_i \subset V_i$ for each $i = 0,1$. Let $G_i = U_i \cup V_i$ for $i = 0,1$. Then, G_0 and G_1 are disjoint open sets in $S(X,D)$, $A_i \subset G_i$ for each $i = 0,1$. The proof is complete. □

For a Tychonoff space X, let $\beta(X)$ denote the Čench-Stone compactification of X.

Example 2.2 (Arhangel'skii and Gordienko[8]) There exists a separable T_2-space X such that $e(X) = 2^c$.

Proof Let $X = \beta(\omega)$. We define another topology on X as follows: every point of ω is isolated and a basic neighbourhood of a point $x \in \beta(\omega)\backslash\omega$ takes the form $\{x\} \cup (U_x \cap \omega)$, where U_x is a neighbourhood of x in $\beta(\omega)$. Clearly, X is T_2 and separable, since ω is a countable dense subspace of X. Since $\beta(\omega)\backslash\omega$ is discrete and closed in X with $|\beta(\omega)\backslash\omega| = 2^c$, we have $e(X) = 2^c$, which completes the proof. □

Remark 1 Arkhangel'skii and Gordienko[8] called the topology of Example 2.2 to be strong topology of $\beta(\omega)$.

Remark 2 Recall from [6] that a space X is absolutely star-Lindelöf if for every open cover $\mathcal{U}$ of X and every dense D of X, there exists a countable subset $F \subset D$ such that $St(F,\mathcal{U}) = X$. Bonanzinga[5] constructed an example having same properties as example 2.2.

It is open whether the extent of a Tychonoff star-Lindel"of space having the property (a) is greater than c. Now, we give a Hausdorff example.

Example 2.3 There exists a T_2 star-Lindelöf space S having the property (a) such that $e(S) = 2^c$.

Proof Let X be the same as in the proof of Example 2.2, $S = S(X,\omega)$. Then, S is a T_2 star-Lindelöf space having the property (a) by Theorem 2.1. Since $\beta(\omega)\backslash\omega$ is discrete closed in S and $|\beta(\omega)\backslash\omega| = 2^c$, we know $e(S) = 2^c$. □

Remark 3 Song[4] constructed that the extent of a T_1 star-Lindelöf space having the property (a) can be arbitrarily large.

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关于 Matveev 的一个问题

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摘 要: 在这篇文章里，我们给出了一个具有性质 (a) Hausdorff star-Lindelöf 空间 X 使得 $e(X) = 2^{\mathfrak{c}}$, 这个例子部分地回答了 Matveev 的一个问题.

关键词: Hasudorff star-Lindelöf 空间；密度.