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Remarks on Two Papers by Y.H. Kim *

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Abstract: Errors and oversights in two papers by Y. H. Kim on some inequalities are pointed out and corrected.

Key words: inequality; power mean; extended mean value.

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In [1], using the properties of the power mean, Kim proves an interesting inequality

$$n\left(\frac{a_1}{n}+\frac{a_2}{n}+\cdots+\frac{a_n}{n}\right)^{x+y} \le n\left(\sum_{i=1}^{n}\frac{1}{n}a_i^x\right)^{\frac{x+y}{x}} \le a_1^{x+y}+a_2^{x+y}+\cdots+a_n^{x+y}$$

for a set of nonnegative real numbers $a_1, a_2, \cdots, a_n$, $n \ge 1$, and $1 \le x, 0 \le y$, with all equalities holding if and only if all a_i are the same.

Besides, some related inequalities are stated in Theorems 2, 3 and 4 of [1]. However, it should be noted that there are some errors in these three theorems. Indeed, the left inequalities of Theorem 3 and Theorem 4 have to be reversed, and the equalities conditions of these three theorems have to be completed. Instead of these three theorems, we have

Theorem A *Let $a_1, a_2, \cdots, a_n$ be a set of n nonnegative real numbers, $n \ge 1$, $0 < x \le 1, -x \le y \le 0$. Then*

$$n\left(\prod_{i=1}^{n}a_i\right)^{\frac{x+y}{n}} \le \sum_{i=1}^{n}a_i^{x+y} \le n\left(\sum_{i=1}^{n}\frac{1}{n}a_i^x\right)^{\frac{x+y}{x}} \le n\left(\sum_{i=1}^{n}\frac{1}{n}a_i\right)^{x+y}$$

with all equalities holding if and only if all a_i are the same or $x+y=0$.

Theorem B *Let $a_1, a_2, \cdots, a_n$ be a set of n nonnegative real numbers, $n \ge 1$, $1 \le x, -2x \le y \le -x$. Then*

$$n\left(\sum_{i=1}^{n}\frac{1}{n}a_i^x\right)^{\frac{x+y}{n}} \le n\left(\sum_{i=1}^{n}\frac{1}{n}a_i\right)^{x+y} \le n\left(\prod_{i=1}^{n}a_i\right)^{\frac{x+y}{n}} \le \sum_{i=1}^{n}a_i^{x+y}$$

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with all equalities holding if and only if all a_i are the same or $x+y=0$.

Theorem C *Let $a_1, a_2, \cdots, a_n$ be a set of n positive real numbers, $n \geq 1$, $1 \leq x, y \leq -2x$. Then*

$$n\left(\sum_{i=1}^{n}\frac{1}{n}a_i^x\right)^{\frac{x+y}{x}} \leq n\left(\sum_{i=1}^{n}\frac{1}{n}a_i\right)^{x+y} \leq n\left(\prod_{i=1}^{n}a_i\right)^{\frac{x+y}{n}} \leq n\left(\sum_{i=1}^{n}\frac{1}{na_i^x}\right)^{\frac{-(x+y)}{x}} \leq \sum_{i=1}^{n}a_i^{x+y}$$

with all equalities holding if and only if all a_i are the same.

In [2], Kim intends to use the monotonicity property of the so-called extended mean values to give a direct and simple proof of the following two interesting and important inequalities

$$\frac{b^{x+y}-a^{x+y}}{b^x-a^x} \geq \frac{x+y}{x}\left(\frac{a+b}{2}\right)^y, \tag{1}$$

where $0<a<b, 1 \leq x$ and $1 \leq y$ are real numbers, and

$$\frac{b^{x+y}-a^{x+y}}{b^x-a^x} \geq \frac{x+y}{x}(ab)^{\frac{y}{2}}, \tag{2}$$

where $0<a<b, 0<x$ and $0<y$ are real numbers.

The so-called extended mean values are defined by the formulation

$$E(r,s;a,b)=\left[\frac{r}{s}\cdot\frac{b^s-a^s}{b^r-a^r}\right]^{\frac{1}{s-r}}, \quad rs(r-s)(a-b)\neq 0$$

$$E(0,0;a,b)=\sqrt{ab}, \quad a\neq b,$$

where a and b are positive numbers and r and s are real numbers.

It is well known that $E(r,s;a,b)$ is increasing in both a and b and in both r and s [3]. The method of using this property to prove (1) and (2) is really simpler than those used in [4] and [5] respectively. But unfortunately, Kim's proof did not use this method correctly. The main errors appear in the Lemma of [2] with the assertions that the inequality

$$\frac{b^s-a^s}{b^r-a^r} \geq \frac{s}{r}(\frac{a+b}{2})^{s-r} \tag{3}$$

holds for $r \leq 1, s \leq 2$ and $s<r$ as well as the inequality (3) reverses for $r \leq 1, s \leq 2$ and $s>r$ are not always true. In fact, $a=1, b=2, r=1$ and $s=-1$ imply that $\frac{b^s-a^s}{b^r-a^r} < \frac{s}{r}(\frac{a+b}{2})^{s-r}$ as well as $a=1, b=16, r=-\frac{1}{2}$ and $s=\frac{1}{4}$ imply that $\frac{b^s-a^s}{b^r-a^r} > \frac{s}{r}(\frac{a+b}{2})^{s-r}$. The reason of causing these errors seemed as if the role of $\mathrm{sgn}(\frac{s}{r})$ is ignored.

Errors in the Lemma lead to corresponding mistakes in the following Theorems 1, 2 and its remark as well as Theorems 4 and 5 in [2]. We give the correction of Lemma in [2] as follows.

Lemma *If $a>0, b>0, a\neq b$, then we have the three sets of inequalities.*

$$\frac{b^s-a^s}{b^r-a^r} \geq \frac{s}{r}(\frac{a+b}{2})^{s-r} \tag{3}$$

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holds for any one of the following cases: (i) $r \geq 1, s \geq 2$ *and* $s > r$*; (ii)* $0 < r \leq 2, 0 < s \leq 1$ *and* $s < r$*; (iii)* $r < 0$ *and* $0 < s \leq 2$*; (iv)* $r < 0, s < 0$ *and* $s < r$*, with equality holding if and only if* $r = 1, s = 2$ *or* $r = 2, s = 1$*. The inequality (3) reverses for any one of the following cases: (i)* $r \geq 2, s \geq 1$ *and* $s < r$*; (ii)* $0 < r \leq 1, 0 < s \leq 2$ *and* $s > r$*; (iii)* $0 < r \leq 2$ *and* $s < 0$*; (iv)* $r < 0, s < 0$ *and* $s > r$*.*

$$\frac{b^s - a^s}{b^r - a^r} > \frac{s}{r}(ab)^{\frac{s-r}{2}} \tag{4}$$

for $r > 0, s > 0$*, and* $s > r$ *or* $r < 0, s < 0$*, and* $s < r$*. The inequality (4) reverses for* $r > 0, s > 0$*, and* $s < r$ *or* $r < 0, s < 0$*, and* $s > r$*.*

$$\frac{s}{r}(ab)^{\frac{s-r}{2}} < \frac{b^s - a^s}{b^r - a^r} \leq \frac{s}{r}\left(\frac{a+b}{2}\right)^{s-r} \tag{5}$$

for $0 < r \leq 1, 0 < s \leq 2$*, and* $s > r$*. The inequality (5) reverses for* $0 < r \leq 2, 0 < s \leq 1$*, and* $s < r$*.*

The lemma is easily deduced from the fact that $E(r, s; a, b)$ is an increasing function in both a and b and in both r and s.

Corrections of the above mentioned theorems in [2] can be easily deduced from the lemma and so are omitted.

Finally, we would like to point out that an overall settlement of studying inequalities (1) and (2) have been made in [6] and [7] respectively by using the Theorem 3 in [8].

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关于 Y.H. Kim 两篇论文的注记

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摘　要: 指出并改正了 Y.H.Kim 两篇论文中关于某些不等式的错误和疏忽.

关键词: 不等式；幂平均；广义幂平均.

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