Journal of Mathematical Research & Exposition
Vol.24, No.1, 49–54, February, 2004

Singular Homoclinic Orbits and Limit Cycles in Positive Second Order Slow-Fast Systems *

XIAO Jian, *SHENG Li-ren*

(Dept. of Math., Anhui University, Hefei 230039, China)

Abstract: In the paper we consider a wide class of slow-fast second order systems and give sufficient conditions for the existence of a singular limit cycle related to a homoclinic orbit.

Key words: homoclinic orbit; limit cycle; singular perturbation; slow manifold.

Classification: AMS(2000) 34C25/CLC number: O175.12

Document code: A **Article ID:** 1000-341X(2004)01-0049-06

1. Introduction and notion

A homoclinic orbit of a continuous time autonomous dynamical system is an orbit tending to an equilibrium in both time direction. Homoclinic orbits play an important role in the analysis of dynamical systems depending upon a parameter α. They are intersection of the stable (incoming) and the unstable (outgoing) invariant manifold of a saddle equilibrium point. Even in the second order case, where the saddle invariant manifolds are one-dimensional, to prove the existence of homoclinic orbits is extremely difficult. Analytical solutions are only available in very special cases (see [1,2]).

At the present time, there is considerable interest in the theory of singular perturbation. Szmolyan has studied the existence of "slow" homoclinic orbits in n-dimensional slow-fast systems. However, these orbits are very special since they do not involve the fast dynamics of the system (see [3]). Kuznetsov, Murator and Rinaldr have discussed the existence of singular homoclinic orbit at α_0, which implies the existence of a singular limit cycle in H.1 systems for $\alpha > \alpha_0$or$\alpha < \alpha_0$(see [1]), but they have not had a discussion on existence conditions of a singular homoclinic orbit and a singular limit cycle for the slow-fast systems.

In the next two sections we extend the above related problems and give existence conditions of a singular limit cycle and a singular homoclinic orbit to a class of positive second order slow-fast systems. As its application, in Section 4 we analyse a special second order dynamical system modeling a three stage food chain (see [1]).

*Received date: 2001-05-14
Foundation item: Supported by Natural Science Foundation of Anhui Education Committee (2000J1008)
Biography: XIAO Jian (1963-), male, Professor.

2. Preliminaries

Consider a positive second order slow-fast dynamical system

$$\begin{cases} \varepsilon\dot{x} = xF(x,y,\alpha), \\ \dot{y} = yf(x,y,\alpha), \end{cases} \tag{1}$$

where ε is a small positive parameter, x and y are the fast and slow variables, respectively, $\alpha \in J = (\alpha_1, \alpha_2)$ is a parameter, and the functions F and G are sufficiently smooth. Systems of this type frequently appear in population dynamics and chemical kinetics. We restrict our attention to the positive octant.

There are two dynamical systems associated with (1).

(I) Fast system

$$\dot{x} = xF(x,y,\alpha) \tag{2}$$

with $x \in R_+$, and $y \in R_+$ as a parameter. Nontrivial orbits of system (2) become parallel lines $y =$ const if imbedded in the plane (x, y).

The equilibria of the fast system (2) for a given value of α are the union of trivial and nontrivial branches

$$M_\alpha = \{(x,y)|x = 0\} \cup \{(x,y)|F(x,y,\alpha) = 0\}. \tag{3}$$

The trivial and nontrivial branches of $M_\alpha$can intersect at some points with $x = 0$ (self-crossing points). The y-axis is invariant for each $\alpha \in J$.

(II) Slow system

$$\dot{y} = yG(x,y,\alpha), \quad (x,y) \in M_\alpha \tag{4}$$

with $x \in R_+$ as a parameter, and $y \in R_+$. The x-axis is invariant for each $\alpha \in J$.

Definition 1 *A singular orbit of (1) is an oriented curve formed by the concatenation of alternate orbits of the slow and fast system.*

Definition 2 *A singular limit cycle is a closed isolated singular orbit.*

Definition 3 *A linearly unstable equilibrium point of the slow system (4) on the stable branch of the slow manifold (3) is called a singular saddle of system (1).*

Denote stable and unstable manifolds of system (1) by $W^{\pm}_{i,\varepsilon}(\alpha), i = 1, 2$, that is the two singular orbits $W^{+}_{1,\varepsilon}(\alpha)$ and $W^{+}_{2,\varepsilon}(\alpha)$ coming out from the saddle form its unstable manifold and the ones $W^{-}_{1,\varepsilon}(\alpha)$ and $W^{-}_{2,\varepsilon}(\alpha)$ coming into the saddle form its stable manifold.

Hereafter, we shall consider slow-fast systems (1) which have a slow manifold M_α with a finite number of critical points $P(a_1(\alpha), a_2(\alpha))$ in positive octant for each $\alpha \in J$ and a finite number of points $S(b(\alpha), 0)$ such that $b > a_1$ and $F(b, 0, \alpha) = 0$ for $\alpha \in J$. Define

$$a_1 = \min\{a|P(a_1(\alpha), a_2(\alpha))\} \text{ and } b_1 = \min\{b|F(b,0,\alpha) = 0, b > a_1\}.$$

Suppose the following conditions hold:

(A_1) $F_{xx} \neq 0$ and $G > 0$ at the critical points;

(A_2) $F_y < 0, F_x > 0$ resp. $F_x < 0$ on $F(x,y,\alpha) = 0$ for $0 \le x < a_1$ (resp. $a_1 < x < b_1$) and $F(0,0,\alpha) > 0$;

(A$_3$) $G_y > 0$ for $0 \le x \le b_1, G(0,y,\alpha) < 0$ for $y \in R_+$ and let the set

$$S = \{(x,y)|F(x,y,\alpha) = G(x,y,\alpha) = 0, \text{ for } a_1 \le x \le b_1\}$$

be empty set or a singleton.

Lemma 1 *Assume that system (1) has a singular saddle $S(\alpha_0)$ for $\alpha = \alpha_0 \in J$. Then*

(i) the system has a saddle $S(\alpha)$ for all αclose to α_0 and all sufficiently small $\varepsilon > 0$;

(ii) there is a neighborhood of the saddle in which the distance between its stable and unstable manifolds $W^{\pm}_{i,\varepsilon}(\alpha)$ and the corresponding manifolds $W^{\pm}_{i,0}(\alpha), i = 1,2$, measured along any local transversal cross-section is a differentiable function of (ε, α) defined for all αclose to α_0 and all sufficiently small $\varepsilon > 0$ and vanishing for $\varepsilon \to 0$.

Proof To prove Lemma 1, we need only checking system (1) is an H.1 system (see [1]).

Condition (A$_1$) implies each critical point is a quadratic fold and not an equilibrium of (1). Condition (A$_2$) implies that the coordinates of such folds are differential with respect to α, meanwhile the curve $N = \{(x,y)|F(x,y,\alpha) = 0, (x,y) \in R^2_+\}$ can be expressed as $y = \varphi(x,y)$ for $(x,y) \in R^2_+$, and all values of $\alpha \in J$. Hence system (1) satisfies H.1 conditions. The Lemma follows immediately from Lemma 1 of [1].

3. Existence theorems of a singular limit cycle and a singular homoclinic orbit

Let U be any neighborhood of the set $\{(x,y)|0 \le x \le b_1, 0 \le y \le a_2\}$. We shall prove

Theorem 1 *In the above situation, if system (1) has a singular saddle $S(\alpha_0)$ for $\alpha = \alpha_0 \in J$, with $S(\alpha_0) = (b_1, 0)$, then*

(1) U contains a unique asymptotic stable singular limit cycle of system (1);

(2) in U, the system has a unique asymptotic stable singular limit cycle for $\alpha > \alpha_0$ or $\alpha < \alpha_0$ with α close to α_0 and all sufficiently small $\varepsilon > 0$.

Proof (1) Under condition (A_2), the system (1) has a unique self-crossing point $R(0, c(\alpha))$ with $0 < c < a_2$ for each $\alpha \in J$. We define

$$Q = \{(x,y)|F(x,y,\alpha) = G(x,y,\alpha) = 0, \quad \alpha \in J, 0 \le x \le a_1\},$$

hence the set is nonempty and bounded. Let $Q_M = \max\{(x,y)|(x,y) \in Q\}$ and $Q_m = \min\{(x,y)|(x,y) \in Q\}$. Then we have $R < Q_m \le Q_M < P$ (see Fig 1.).

In the sequel, PS denotes the segment P from to S. For small $\varepsilon > 0$ the following is true. For $0 < x < x_m$(resp. $x_M < x < b_1$) the curve $F(x,y,\alpha_0) = 0$ lies below (resp. above) the curve$G(x,y,\alpha_0) = 0$. Along $F(x,y,\alpha_0) = 0, \dot{x} = 0$, and $\dot{y} > 0$ if $x_M < x < b_1$, $\dot{y} < 0$ if $0 < x < x_m$. The curve PS and the y-axis for $y > c$ are the stable branches of the slow manifold M_{α_0}, the curve PR and y-axis for $y < c$ are the unstable branches of the slow manifold M_{α_0}, and points Q lie on the unstable branch of the same manifold.

For $\delta > 0$ sufficiently small, the orbit of system (1) starting from $D_1(\delta, c)$ follows the line $y = c$ to the right until it meets the curve $F(x,y,\alpha_0) = 0$; follows $F(x,y,\alpha_0) = 0$ to the vertex $P(a_1(\alpha_0), a_2(\alpha_0))$. Since this point is a quadratic fold of the slow manifold M_{α_0}, the singular orbit continues horizontally from P to $P'(0, a_2(\alpha_0))$ on the y-axis and then follows the y-axis to point $P''(0, a_3(\alpha_0))$ with P'' on the unstable branch of the slow manifoldM_{α_0}. The point satisfies the following properties:

(i) $\int_{a_2}^{a_3} \frac{F(0,y,\alpha_0)}{yG(0,y,\alpha_0)} dy = 0$ (see [1]);

(ii) $a_3 < c$.

We shall prove (ii). If not, then $a_3 = c$. A simple similar analysis shows that the self-crossing point $R(0, c)$ is an equilibrium point. According to (A_3), the point is not an equilibrium point, a contradiction. This prove (ii). The orbit follows the line $y = a_3$ to point (δ, a_3).

Similarly, the unstable separatrix $W^+_{1,\varepsilon}$ of the singular saddle $S(\alpha_0)$ intersects the line $x = \delta$ at point $D_2(\delta, d)$ with $a_3 > d$.

Therefore, there exists a continuous function $H(\varepsilon, \alpha_0) : D_1D_2 \mapsto D_1D_2$. Since $H(\varepsilon, \alpha_0)$ maps D_1D_2 to itself, it has a fixed point. This proves that the system has a closed singular orbit Γ. (See Fig 2)

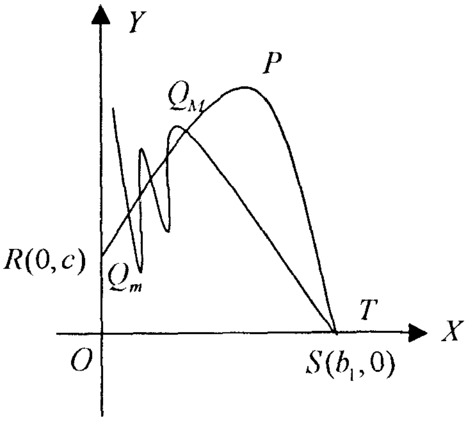

Fig 1

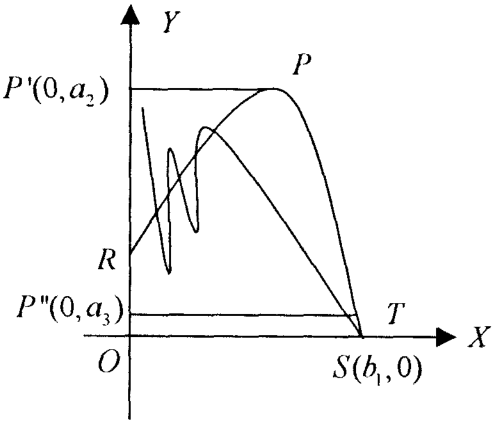

Fig 2

We compute

$$
\begin{aligned}
I = & \int_\Gamma \mathrm{div}(xF, yG)\mathrm{d}t = \frac{1}{\varepsilon}\int_\Gamma (F + xF_x)\mathrm{d}t + \int_\Gamma (G + yG_y)\mathrm{d}t \\
= & \frac{1}{\varepsilon}[\int_{P'P''} (F + xF_x)\mathrm{d}t + \int_{P''T} (F + xF_x)\mathrm{d}t + \int_{TP} (F + xF_x)\mathrm{d}t + \int_{PP'} (F + xF_x)\mathrm{d}t] + \\
& \int_\Gamma (G + yG_y)\mathrm{d}t \\
= & \frac{1}{\varepsilon}(I_1 + I_2 + I_3 + I_4) + \int_\Gamma (G + yG_y)\mathrm{d}t,
\end{aligned}
$$

$$
I_1 = \int_{P'P''} (F + xF_x)\mathrm{d}t = \int_{P'P''} \frac{F + xF_x}{yG}\mathrm{d}y \approx \int_{a_2}^{a_3} \frac{F(0, y, \alpha_0)}{yG(0, y, \alpha_0)}\mathrm{d}y = 0.
$$

Since the functions F and F_x are continuous functions and t is sufficiently small on $P''T$, we have $I_2 \approx 0$. Similarly $I_4 \approx 0$, and

$$
I_3 = \int_{TP}(F + xF_x)\mathrm{d}t = \int_{TP} \frac{F + xF_x}{yG}\mathrm{d}y \approx \int_{TP} \frac{xF_x}{yG}\mathrm{d}y < 0,
$$

hence $I < 0$. This proves that the closed orbit Γ is asymptotic stable (see [4,5]).

We shall show that system (1) has a unique limit cycle. If not, let Γ_1 and Γ_2 be two closed orbits such that $D(\Gamma_1) \subset D(\Gamma_2)$. Let $D(\Gamma_2)\backslash D(\Gamma_1)$ be the region bounded by Γ_1 and Γ_2. Then the system has not equilibria in $D(\Gamma_2)\backslash D(\Gamma_1)$. On the other hand, $\Gamma_i (i = 1, 2)$, is an asymptotic stable closed orbit, a contradiction (see [4,5])). This proves that Γ is a unique asymptotic stable singular limit cycle, and is in U.

To prove (2) of theorem 1, we shall use the Lemma 1. The Lemma shows that the system has a saddle $S(\alpha)$ for all α close to α_0 and all sufficiently small $\varepsilon > 0$. Further there is a neighborhood of the saddle in which the distance between its stable and unstable manifold $W^{\pm}_{i,\varepsilon}(\alpha)$ and the corresponding manifold $W^{\pm}_{i,0}(\alpha), i = 1, 2$, measured along any local transversal cross-section is a differentiable function of (ε, α) defined for all α close to α_0 and all sufficiently small $\varepsilon > 0$ and vanishing for $\varepsilon \to 0$. Hence a similar analysis proves (2) of Theorem 1.

For $\delta > 0$ sufficiently small, denote the first intersection of $W^{+}_{1,\varepsilon}$ with the line $x = \delta$ by $W(\delta, d(\alpha))$ and a saddle $S(\alpha)$ by $(\tilde{x}(\alpha), \tilde{y}(\alpha))$, and the self-crossing point $R(0, c)$ by $(0, c(\alpha))$. We shall prove the following Theorem 2.

Theorem 2 *In the above Theorem 1 situation, if there is a value of $\alpha = \tilde{\alpha}_0 \in J$ such that $c(\tilde{\alpha}_0) \le \tilde{y}(\tilde{\alpha}_0)$, then the system has a singular homoclinic orbit.*

Proof Define a function $\Delta(\alpha, \delta) = \tilde{y}(\alpha) - d(\alpha)$, then it is independent on δ. By Theorem 1, we have

$$\Delta(\alpha_0, \delta) = \tilde{y}(\alpha_0) - d(\alpha_0) = 0 - d(\alpha_0) < 0,$$

$$\Delta(\tilde{\alpha}_0, \delta) = \tilde{y}(\tilde{\alpha}_0) - d(\tilde{\alpha}_0) \ge c(\tilde{\alpha}_0) - d(\tilde{\alpha}_0) > 0.$$

On the other hand, the function F and G are sufficiently and the coordinates of these orbits of system (1) are continuous with respect to α, then $\Delta(\alpha, \delta)$ is a continuous function. Hence there exists a value of $\alpha = \hat{\alpha} \in J$ such that $\Delta(\hat{\alpha}, \delta) = 0$. This proves that the system has a singular homoclinic orbit.

4. Applications

Hereafter we consider the following predator-prey system

$$\begin{cases} \varepsilon \dot{x} = x\left[r(1 - \frac{x}{K}) - \frac{ay}{b+x}\right] = xF(x, y), \\ \dot{y} = y\left[\frac{aex}{b+x} - m - \frac{\alpha}{d+y}\right] = yG(x, y, \alpha), \end{cases} \tag{5}$$

where x and y are densities of prey and predator populations, respectively, $\varepsilon, r, K, a, b, d, e, m$ are positive parameters and α is the density of the super predator measured in suitable units. These parameters are of biological significance [1].

If $ae - m > 0, K > \left(\frac{ae+m}{ae-m}\right) b$ and $\frac{[(ae-m)K-(ae+m)b]\cdot[4adK+r(b+K)^2]}{4adK[(ae-m)K-bm]} > 1$, then the system has a unique asymptotic stable singular limit cycle. In order to deal with system (5), we have to check the assumptions of Theorem 1. The conditions (A_1)-(A_3) in Section 2 are satisfied by the system.

The condition (A_1) holds, since the critical point $P(a_1, a_2) = \left(\frac{K-b}{2}, \frac{r(b+K)^2}{4aK}\right)$, the self-crossing point $R(0,c)=\left(0,\frac{br}{a}\right)$, $F_{xx}(P)=\frac{-4r}{K(b+K)}<0$ and

$$G(P)=\frac{(ae-m)K-(ae+m)b}{b+K}-\frac{4aK\alpha}{4adK+r(b+K)^2}>0.$$

The condition (A_2) holds, since $S(K,0)$ is a singular saddle point for $\alpha=\alpha_0$ with $\alpha_0=\frac{d[(ae-m)K-bm]}{b+K}>0$, $F_y=\frac{-a}{b+x}<0$ for $(x,y)\in R_+^2$, $F(0,0,\alpha)=r>0$ and $F_x=\frac{r(K-b-2x)}{K(b+x)}>0$(resp. <0) for $0\le x<\frac{K-b}{2}$(resp. $\frac{K-b}{2}<x<K$) on $F(x,y,\alpha_0)=0$. The condition (A_3) holds, since $G_y=\frac{\alpha}{(d+y)^2}>0$ for $(x,y)\in R_+^2$, $G(0,y,\alpha)=-m-\frac{\alpha}{d+y}<0$, meanwhile the curve $G(x,y,\alpha)=0$ can be expressed as

$$y=\frac{\alpha(b+x)}{(ae-m)(b+x)-abe}-d\quad\text{for}\quad(x,y)\in R_+^2$$

and with

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-abe\alpha}{[(ae-m)(b+x)-abe]^2}<0,$$

hence the set $\{(x,y)|F(x,y)=G(x,y,\alpha)=0, 0<x<\frac{K-b}{2}\}$ is a singleton and the set $\{(x,y)|F(x,y)=G(x,y,\alpha)=0,\frac{K-b}{2}<x\le K\}$ is empty or a singleton. This shows that the system has a unique asymptotic stable singular limit cycle.

Further with the addition of the following condition

$$\frac{[(ae-m)K-(ae+m)b]\cdot[4adK+r(b+K)^2]}{4(b+K)(br+ad)[(ae-m)K-abe]}>1,$$

hence there exists $\alpha=\tilde{\alpha}_0.=\frac{(br+ad)[(ae-m)K-abe]}{aK}$ such that $c(\tilde{\alpha}_0)=\tilde{y}(\tilde{\alpha}_0)$, which implies Theorem 2 holds. The system has a singular homoclinic orbit. This result coincides with Theorem 3 of [1].

References:

[1] KUZNETSOV YU A, et al. *Homoclinic bifurcations in slow-fast second order system* [J]. Nonlinear Anal., 1995, **25**: 747–762.

[2] GUCKENHEIMER J, HOLMES P H. *Dynamical Systems and Bifurcations of Vector Fields* [M]. Springer-Verlag, 1986.

[3] SZMOLYAN P. *Transversal heteroclinic orbits in singular perturbation problems* [J]. J. differential Equations, 1991, **92**: 252–281.

[4] Hale J K. *Ordinary Differential Equations* [M]. 2nd ed., Krieger, Malabar, Florida, 1980.

[5] YE Yan-qian. et al. *Theory of Limit Cycle* [M]. Trans. Math. Monographs, 66, Amer. Math. Soc., 1986.

正二阶快 - 慢系统中的奇性同宿轨道和极限环

肖 箭， 盛 立 人

(安徽大学数学系，安徽 合肥 230039)

摘 要: 本文研究一类正二阶快 - 慢系统中奇性同宿轨道和极限环，并且给出了此系统存在奇性同宿轨道和极限环的充分条件.

关键词: 同宿轨道；极限环；奇摄动；慢流形.