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# The Law of the Iterated Logarithm for Independent Random Variables with Multidimensional Parameters and Its Application *

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**Abstract:** For a set of i.i.d.r.v. indexed by positive integer $d$-dimensional lattice points, and for some general normalizing sequence, we determine necessary and sufficient conditions for the law of iterated logarithm. As its application, we give conditions for the existence of moments of the supremum of normed partial sums.

**Key words:** Law of the iterated logarithm; multidimensional parameter; moment of supremum.

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## 1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and suppose all random variables are defined on this space.

Let $d \geq 1$ be an integer and $N^d$ denote the positive integer $d$-dimensional lattice points with coordinate-wise partial ordering, $\leq$, i.e. for every $\overline{m} = (m_1, \cdots, m_d)$, $\overline{n} = (n_1, \cdots, n_d) \in N^d$ $\overline{m} \leq \overline{n}$ if and only if $m_i \leq n_i$, $i = 1, \cdots, d$. For $\overline{n} \in N^d$, define the product $|\overline{n}| = \prod_{i=1}^d n_i$. As we know, in the case $d > 1$ the term $\overline{n} \to \infty$ could be understood by some different meaning. In some papers the limit $\overline{n} \to \infty$ means that $n_i \to \infty$ for all $i = 1, \cdots, d$ (or equivalently $\min_i n_i \to \infty$). In this paper the limit $\overline{n} \to \infty$ is interpreted or $|\overline{n}| \to \infty$ (or equivalently $\max_i n_i \to \infty$). By the way, we remark that the limit $\max_i n_i \to \infty$ induces a finer topology then the limit $\min_i n_i \to \infty$.

Let $\{X, X_n, X_{\overline{n}}, n \geq 1, \overline{n} \in N^d\}$ be a family of independent identically distributed random variables(i.i.d.r.v.). For $n \geq 1$ and $\overline{n} \in N^d$, define

$$S_n = \sum_{m=1}^{n} X_m, \quad S_{\overline{n}} = \sum_{\overline{m} \leq \overline{n}} X_{\overline{m}} = \sum_{m_1=1}^{n_1} \cdots \sum_{m_d=1}^{n_d} X_{(m_1, \cdots, m_d)}.$$

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In order to bring into focus the essentials of this paper, we begin with a description Wichura's work[1], i.e.

$$\limsup_{\min_i\to\infty}\frac{|S_{\overline{n}}|}{\sqrt{2|\overline{n}|L_2(|\overline{n}|)}}<\infty \quad \text{a.s.} \tag{1.1}$$

if and only if

$$EX=0,\quad EX^2<\infty \text{ and } EX^2(L(|X|))^{d-1}(L_2(|X|))^{-1}<\infty, \tag{1.2}$$

here and throughtout this paper $L(x)=\log(\max\{e,x\})$ and $L_2(x)=L(L(x))$, $x>0$. By the results of Li and Wu[2] for $d$-dimensional($d>1$), $B$-valued i.i.d.r.v., result (1.1) can be strenghtened to

$$\limsup_{\overline{n}\to\infty}\frac{|S_{\overline{n}}|}{\sqrt{2|\overline{n}|L_2(|\overline{n}|)}}<\infty \quad \text{a.s.} \tag{1.3}$$

We shall consider the constant sequence $\{a_{\overline{n}},\overline{n}\in N^d\}$ of the type

$$a_{\overline{n}}=n_1^{1/t_1}\cdots n_d^{1/t_d},\quad t_i\in(0,2],\quad i=1,\cdots,d. \tag{1.4}$$

One of the main results of the paper deals with the law of the iterated logarithm for the normalizing sequence $\sqrt{2a_{\overline{n}}^2L_2\left(a_{\overline{n}}^2\right)},\overline{n}\in N^d$. On the Marcinkiewicz-Zugmund strong law of large number for the normalizing sequence $a_{\overline{n}},\overline{n}\in N^d$ of the type (1.4) one can see [3], [4] and [5] etc.

The others give necessary and sufficient conditions for the existence of moments of the supremum of normed partial sums $\sup_{\overline{n}\in N^d}\left(\sqrt{a_{\overline{n}}^2L_2\left(a_{\overline{n}}^2\right)}\right)^{-1}|S_{\overline{n}}|$ as the application of the law of the iterated logarithm , i.e. the first main result.

The simplest case $d=1$, $a_n=\sqrt{n}$ has been investigated fisrt by Siegmund[6] and Teicher[7]. They proved that if $EX=0$ and $p\geq 2$, then the following statement are equivalent:

$$\begin{cases} EX^2(L(|X|))(L_2(|X|))^{-1}<\infty & \text{if } p=2,\\ E|X|^p<\infty & \text{if } p>2,\end{cases} \tag{1.5}$$

$$E\sup_{n\geq 1}\left|\frac{X_n}{\sqrt{nL_2(n)}}\right|^p<\infty, \tag{1.6}$$

$$E\sup_{n\geq 1}\left|\frac{S_n}{\sqrt{nL_2(n)}}\right|^p<\infty. \tag{1.7}$$

Siegmund[6] proving it for integers $2,(3,4,\cdots)$ and Teicher[7] for $p>2$.

For the case $d>1$, let $a_{\overline{n}}=\sqrt{|\overline{n}|}$, $EX=0$ and $p\geq 2$, Gut[8] established that

$$\begin{cases} EX^2(L(|X|))^d(L_2(|X|))^{-1}<\infty & \text{if } p=2,\\ E|X|^p<\infty & \text{if } p>2,\end{cases} \tag{1.8}$$

$$E\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{|\overline{n}|L_2(|\overline{n}|)}}\right|^p<\infty, \tag{1.9}$$

$$E\sup_{\overline{n}\in N^d}|\frac{S_{\overline{n}}}{\sqrt{|\overline{n}|L_2(|\overline{n}|)}}|^p<\infty \tag{1.10}$$

are all equivalent.

The key to the proof of Gut's result[8] is the law of the iterated logarithm(see [1]).

The results of the moments of the supremum of normed partial sums related to the Marcinkiewicz-Zygmund strong law of large numbers for the multidimensional parameters can be found in [8], [9] etc.

The proof of the main results of this paper will be based on the following lemmas.

**Lemma 1.1** (see[4] Lemma 1.1 or [9] Lemma 1.1) *Let $\alpha_1,\cdots,\alpha_d$ be real numbers, $\alpha=\max\{\alpha_i,i=1,\cdots,d\}$, $t_1,\cdots,t_d>0$. Put $p=\max\{\alpha_i t_1,\cdots,\alpha_d t_d\}$, $q=\operatorname{card}\{i:\alpha_i t_i=p,i=1,\cdots,d\}$, $r=\operatorname{card}\{i,\alpha_i=0,i=1,\cdots,d\}$. For each $x>0$, put*

$$f(x)=\sum_{n_i^{1/t_1}\cdots n_d^{1/t_d}\le x} n_1^{\alpha_1-1}\cdots n_d^{\alpha_d-1}.$$

*Then there exist constants $c_1,c_2>0$ such that for every $x>x_0$:*

*(i) If $\alpha\le 0$, then*

$$c_1(L(x))^r\le f(x)\le c_2(L(x))^r.$$

*(ii) If $\alpha>0$, then*

$$c_1x^p(L(x))^{q+r-1}\le f(x)\le c_2x^p(L(x))^{q+r-1}.$$

**Lemma 1.2** *Let $0<t_1,\cdots,t_d\le 2$, $\max\{t_i,i=1,\cdots,d\}=2$, $r=\operatorname{card}\{i:t_i=2,i=1,\cdots,d\}$. Let $a_{\overline{n}}$ be of the type (1.4), $d_i=\operatorname{card}\{\overline{n}:i<a_{\overline{n}}^2\le i+1\}$ and $p>0$. Then for some $c_3>0$,*

$$\sum_{i=1}^{j}(iL_2(i))^{-\frac{p}{2}}d_i\le\begin{cases}c_3j^{-\frac{p}{2}+1}(L(j))^{r-1}(L_2(j))^{-\frac{p}{2}} & \text{if } p\ne 2,\\ c_3(L(j))^r(L_2(j))^{-1} & \text{if } p=2.\end{cases}$$

**Proof** By Abel method and Lemma 1.1, we have

$$\begin{aligned}&\sum_{i=1}^{j}(iL_2(i))^{-\frac{p}{2}}d_i\\ &=(jL_2(j))^{-\frac{p}{2}}\sum_{i=1}^{j}d_i+\sum_{i=1}^{j-1}\left((iL_2(i))^{-\frac{p}{2}}-((i+1)L_2(i+1))^{-\frac{p}{2}}\right)\sum_{k=1}^{i}d_k\\ &\le c_2(jL_2(j))^{-\frac{p}{2}}j(L(j))^{r-1}+2^{\frac{p}{2}}c_2\sum_{i=1}^{j-1}(L(i))^{r-1}(iL_2(i))^{-\frac{p}{2}}\\ &\le\begin{cases}c_3j^{-\frac{p}{2}+1}(L(j))^{r-1}(L_2(j))^{-\frac{p}{2}} & \text{if } p\ne 2,\\ c_3(L(j))^r(L_2(j))^{-1} & \text{if } p=2.\end{cases}\end{aligned}$$

**Lemma 1.3** (see[9] Lemma 1.2) *Let $B$ be a Banach space with norm $\|\cdot\|$ and $\{Y_{\overline{n}}, \overline{n} \in N^d\}$ be independent $B$-valued random variables. Further let $\{a_{\overline{n}}, \overline{n} \in N^d\}$ be a set of positive real numbers such that $a_{\overline{m}} \le a_{\overline{n}}$ if $\overline{m} \le \overline{n}$. Set*

$$U_{\overline{n}} = a_{\overline{n}}^{-1} \sum_{\overline{m} \le \overline{n}} Y_{\overline{m}}, \ V = \sup_{\overline{n} \in N^d} \|U_{\overline{n}}\|, \ W = \sup_{\overline{n} \in N^d} \|a_{\overline{n}}^{-1} Y_{\overline{n}}\|$$

*and suppose that $V < \infty$ a.s.. Then $W < \infty$ a.s. and if $EW^p < \infty$, then $EV^p < \infty$.*

**Lemma 1.4** *Let $0 < t_1, \cdots, t_d \le 2$, $\max\{t_i, i = 1, \cdots, d\} = 2$, $r = \mathrm{card}\{i : t_i = 2, i = 1, \cdots, d\}$.Let $X$ be a real valued with $EX^2(L(|X|))^{r-1}(L_2(|X|))^{-1} < \infty$,and $a_{\overline{n}}$ be of the type (1.4). Then for some $c_4, c_5 > 0$*

$$c_4 \sum_{\overline{n} \in N^d} P\left(|X| \ge \sqrt{a_{\overline{n}}^2 L_2(a_{\overline{n}}^2)}\right) \le EX^2(L(|X|))^{r-1}(L_2(|X|))^{-1}$$

$$\le c_5 \sum_{\overline{n} \in N^d} P\left(|X| \ge \sqrt{a_{\overline{n}}^2 L_2(a_{\overline{n}}^2)}\right)$$

**Proof** By Lemma 1.1 directly, we have the desired result.

## 2. Results and proofs

In this section we state and prove the main results of this paper.

**Theorem 2.1** *Let $\{X, X_{\overline{n}}, \overline{n} \in N^d\}$ be a family of i.i.d.r.v., $0 < t_1, \cdots, t_d \le 2$, $\max\{t_i, i = 1, \cdots, d\} = 2$, $r = \mathrm{card}\{i : t_i = 2, i = 1, \cdots, d\}$. Let $a_{\overline{n}}$ be of the type (1.4). Then*

$$\limsup_{\overline{n} \to \infty} \frac{|S_{\overline{n}}|}{\sqrt{2a_{\overline{n}}^2 \log\log a_{\overline{n}}^2}} < \infty \quad \text{a.s.} \tag{2.1}$$

*if and only if*

$$EX = 0, \ EX^2 < \infty \text{ and } EX^2(\log|X|)^{r-1}(\log\log|X|)^{-1} < \infty. \tag{2.2}$$

**Proof** (sufficiency) It is easy to show that $(\sqrt{2a_{\overline{n}}^2 L_2(a_{\overline{n}}^2)})^{-1}|S_{\overline{n}}| \to 0$ in probability, so by the standard symmetrization procedure, it suffices to prove the theorem under the assumption that $X$ is symmetric.

Let $\tau > 0$, and define $Z_{\overline{n}} = X_{\overline{n}} I(|X_{\overline{n}}| > \tau\sqrt{a_{\overline{n}}^2 / L_2(a_{\overline{n}}^2)})$. First we show that

$$\limsup_{\overline{n} \to \infty} \frac{|\sum_{\overline{m} \le \overline{n}} Z_{\overline{m}}|}{\sqrt{2a_{\overline{n}}^2 L_2\left(a_{\overline{n}}^2\right)}} = 0 \quad \text{a.s.} \tag{2.3}$$

When $r = 1$, by the same argument as [10], it is enough to prove that

$$\sum_{\overline{n} \in N^d} \frac{E|Z_{\overline{n}}|}{\sqrt{a_{\overline{n}}^2 L_2\left(a_{\overline{n}}^2\right)}} < \infty. \tag{2.4}$$

To prove (2.4), let $b_i = \sqrt{i/L_2(i+1)}$, $d_i = \text{card}\{\overline{n} : i < a_{\overline{n}}^2 \leq i+1\}$. We have by Lemma 1.2

$$
\begin{aligned}
&\sum_{\overline{n}\in N^d} \left(\sqrt{a_{\overline{n}}^2 L_2\left(a_{\overline{n}}^2\right)}\right)^{-1} E|Z_{\overline{n}}| \\
&\leq \sum_{i=1}^{\infty}(\sqrt{iL_2(i)})^{-1} d_i E|X|I(|X| > \tau b_i) \\
&= \sum_{i=1}^{\infty}(\sqrt{iL_2(i)})^{-1} d_i \sum_{j=i}^{\infty} E|X|I(\tau b_j < |X| \leq \tau b_{j+1}) \\
&= \sum_{j=1}^{\infty} E|X|I(\tau b_j < |X| \leq \tau b_{j+1}) \sum_{i=1}^{j}(\sqrt{iL_2(i)})^{-1} d_i \\
&\leq c\sum_{j=1}^{\infty} E|X|I(\tau b_j < |X| \leq \tau b_{j+1}) j^{1/2}(L_2(j))^{-1/2} \\
&\leq c\sum_{j=1}^{\infty} EX^2 I(\tau b_j < |X| \leq \tau b_{j+1}) \\
&\leq cEX^2 < \infty.
\end{aligned}
$$

When $r > 1$, proceeding as Lemma 5 and Lemma 6 in [2], we have (2.3).

The remaining of the proof is similar to Theorem 1 of [2], so we omit it.

(necessity)Suppose first that $r = 1$. Without loss of generality, set $t_1 = 2$. Obviously

$$\limsup_{n_1\to\infty} \frac{|\sum_{m_1=1}^{n_1} X_{(m_1,1,\cdots,1)}|}{\sqrt{2n_1 L_2(n_1)}} \leq \limsup_{\overline{n}\to\infty} \frac{|S_{\overline{n}}|}{\sqrt{2a_{\overline{n}}^2 L_2(a_{\overline{n}}^2)}} < \infty \quad \text{a.s.}$$

Then By the result of Strassen[11], we have $EX = 0$ and $EX^2 < \infty$.

For $r > 1$. We also set $t_1 = t_2 = \cdots = t_r = 2$. Then

$$\limsup_{\prod_{i=1}^{r} n_i\to\infty} \frac{|\sum_{m_1=1}^{n_1}\cdots\sum_{m_r=1}^{n_r} X_{(m_1,\cdots,m_r,1,\cdots,1)}|}{\sqrt{2\prod_{i=1}^{r} n_i L_2(\prod_{i=1}^{r} n_i)}} \leq \limsup_{\overline{n}\to\infty} \frac{|S_{\overline{n}}|}{\sqrt{2a_{\overline{n}}^2 L_2(a_{\overline{n}}^2)}} < \infty \quad \text{a.s.}$$

By the result of Wichura[1], $EX = 0$ and $EX^2(L(|X|))^{r-1}(L_2(|X|))^{-1} < \infty$ hold.

**Remark 2.1** When $r = 1$, the sufficiency can also be obtained by Theorem 1 of [12] directly.

**Theorem 2.2** *Let $\{X, X_{\overline{n}}, \overline{n} \in N^d\}$ be a family of i.i.d.r.v., $0 < t_1, \cdots, t_d \leq 2, \max\{t_i, i = 1, \cdots, d\} = 2$, $r = \text{card}\{i : t_i = 2, i = 1, \cdots, d\}$. Let $a_{\overline{n}}$ be of the type (1.4). Let $EX = 0$ and $p \geq 2$, then the following statements are equivalent:*

$$\begin{cases} EX^2(L(|X|))^r(L_2(|X|))^{-1} < \infty & \text{if } p = 2, \\ E|X|^p < \infty & \text{if } p > 2, \end{cases} \tag{2.5}$$

$$E\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p<\infty, \tag{2.6}$$

$$E\sup_{\overline{n}\in N^d}\left|\frac{S_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p<\infty. \tag{2.7}$$

**Proof** (2.5) $\Rightarrow$ (2.6) Define

$$X'_{\overline{n}}=X_{\overline{n}}I(|X_{\overline{n}}|\le\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}),\quad X''_{\overline{n}}=X_{\overline{n}}-X'_{\overline{n}}.$$

It is evident that

$$E\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p\le E\sup_{\overline{n}\in N^d}\left|\frac{X'_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p+E\sup_{\overline{n}\in N^d}\left|\frac{X''_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p. \tag{2.8}$$

Obviously,

$$E\sup_{\overline{n}\in N^d}\left|\frac{X'_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p\le 1.$$

We estimate now the second term on the right-hand side of (2.8). Let $b_i=\sqrt{iL_2(i)}$ and $d_i=\operatorname{card}\{\overline{n}: i<a_{\overline{n}}^2\le i+1\}$. We have by Lemma 1.2

$$\begin{aligned}
&E\sup_{\overline{n}\in N^d}\left|\frac{X''_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p\le\sum_{\overline{n}\in N^d}E\left|\frac{X''_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p\\
&\le\sum_{i=1}^{\infty}d_ib_i^{-p}E|X|^pI(|X|>b_i)\\
&=\sum_{i=1}^{\infty}d_ib_i^{-p}\sum_{j=i}^{\infty}E|X|^pI(b_j<|X|\le b_{j+1})\\
&=\sum_{j=1}^{\infty}E|X|^pI(b_j<|X|\le b_{j+1})\sum_{i=1}^{j}d_ib_i^{-p}\\
&\le\begin{cases} c\sum_{j=1}^{\infty}E|X|^pI(b_j<|X|\le b_{j+1})(L(j))^r(L_2(j))^{-1} & \text{if } p=2\\ c\sum_{j=1}^{\infty}E|X|^pI(b_j<|X|\le b_{j+1}) & \text{if } p>2\end{cases}\\
&\le\begin{cases} cEX^2(L(|X|))^r(L_2(|X|))^{-1} & \text{if } p=2\\ cE|X|^p & \text{if } p>2\end{cases}\\
&<\infty.
\end{aligned}$$

This terminates the first step of the proof.

(2.6) $\Rightarrow$ (2.5) Set $t_1=\cdots=t_r=2$. Since

$$E\sup_{(n_1,\cdots,n_r,1,\cdots,1)\in N^d}\left|\frac{X_{(n_1,\cdots,n_r,1,\cdots,1)}}{\sqrt{\prod_{i=1}^r n_iL_2(\prod_{i=1}^r n_i)}}\right|^p\le E\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p<\infty$$

when $r=1$, by the results of Siegmund[1] and Teicher[2], we have $EX^2L(|X|)(L_2(|X|))^{-1} < \infty$. When $r>1$, by Theorem 3.1 of Gut[3], $EX^2(L(|X|))^r(L_2(|X|))^{-1} < \infty$ holds.

The proof of (2.6) $\Leftrightarrow$ (2.7) is an appropriate modification of that given in [8] for Theorem 3.1.

(2.6) $\Rightarrow$ (2.7) Because of (2.5) the law of the iterated logarithm is true by Theorem 2.1 . It follows that $V=\sup_{\overline{n}}(\sqrt{a^2_{\overline{n}\in N^d}L_2(a^2_{\overline{n}})})^{-1}|S_{\overline{n}}| < \infty$ a.s. since (2.6) holds, that is $EW^p < \infty$, where $W=\sup_{\overline{n}\in N^d}(\sqrt{a^2_{\overline{n}}L_2(a^2_{\overline{n}})})^{-1}|X_{\overline{n}}|$, An application of Lemma 1.3 yields $EV^p < \infty$, i.e. (2.7).

(2.7) $\Rightarrow$ (2.6). Immediate, because $W \le 2^dV$. The theorem is completely proved.

**Theorem 2.3** *Let $\{X, X_{\overline{n}}, \overline{n}\in N^d\}$ be a family of i.i.d.r.v., $0<t_1,\cdots,t_d\le 2$, $\max\{t_i, i=1,\cdots,d\}=2$, $r=\operatorname{card}\{i: t_i=2, i=1,\cdots,d\}$. Let $a_{\overline{n}}$ be of the type (1.4). Let $EX=0$ and $0<p<2$.*

*(i) If $r=1$, then*

$$EX^2(L_2(|X|))^{-1} < \infty, \tag{2.9}$$

$$E\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a^2_{\overline{n}}L_2(a^2_{\overline{n}})}}\right|^p < \infty, \tag{2.10}$$

*are equivalent, and*

$$EX^2 < \infty, \tag{2.11}$$

$$E\sup_{\overline{n}\in N^d}\left|\frac{S_{\overline{n}}}{\sqrt{a^2_{\overline{n}}L_2(a^2_{\overline{n}})}}\right|^p < \infty, \tag{2.12}$$

*are eqivalent.*

*(ii) If $r>1$, then the following statements are eqivalent:*

$$EX^2(L(|X|))^{r-1}(L_2(|X|))^{-1} < \infty, \tag{2.13}$$

$$E\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a^2_{\overline{n}}L_2(a^2_{\overline{n}})}}\right|^p < \infty, \tag{2.14}$$

$$E\sup_{\overline{n}\in N^d}\left|\frac{S_{\overline{n}}}{\sqrt{a^2_{\overline{n}}L_2(a^2_{\overline{n}})}}\right|^p < \infty. \tag{2.15}$$

**Proof** Define

$$X'_{\overline{n}} = X_{\overline{n}}I\left(|X_{\overline{n}}| \le \sqrt{a^2_{\overline{n}}L_2(a^2_{\overline{n}})}\right), \quad X''_{\overline{n}} = X_{\overline{n}} - X'_{\overline{n}}$$

Firat we prove that

$$E\sup_{\overline{n}\in N^d}\left|\frac{X''_{\overline{n}}}{\sqrt{a^2_{\overline{n}}L_2(a^2_{\overline{n}})}}\right|^p \le cEX^2(L(|X|))^{r-1}(L_2(|X|))^{-1}. \tag{1.16}$$

In fact, by Lemma 1.2 and by the same argument as Theorem 2.2, (2.16) holds.

(2.9) $\Rightarrow$ (2.10) It is evident by (2.16) that

$$E\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p \le E\sup_{\overline{n}\in N^d}\left|\frac{X'_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p + E\sup_{\overline{n}\in N^d}\left|\frac{X''_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p$$
$$\le 1 + cEX^2(L(|X|))^{r-1}(L_2(|X|))^{-1} < \infty.$$

(2.10) $\Rightarrow$ (2.9) If $EX^2(L_2(|X|))^{-1} = \infty$, then by Lemma 1.4

$$\sum_{\overline{n}\in N^d} P(|X| > M\sqrt{a_{\overline{n}}^2L_2(|X|)}) = \sum_{\overline{n}\in N^d} P\left(\left|\frac{X_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p > M^p\right)$$
$$= \infty \quad \text{for any} \quad M > 0.$$

The Borel-Cantelli lemma implies that $\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p > M^p$ a.s., thus

$$\sup_{\overline{n}\in N^d}\left|\frac{X_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right|^p = \infty \quad \text{a.s.}$$

(2.11) $\Rightarrow$ (2.12) If $EX^2 < \infty$, then we have (2.10) by (2.9)$\Rightarrow$(2.10) and

$$\sup_{\overline{n}\in N^d}\left|\frac{S_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right| < \infty \quad \text{a.s.}$$

by Theorem 2.1, hence by Lemma 1.3, (2.12) holds.

(2.12) $\Rightarrow$ (2.11) If $EX^2 = \infty$, then by Theorem 2.1 $\sup_{\overline{n}\in N^d}\left|\frac{S_{\overline{n}}}{\sqrt{a_{\overline{n}}^2L_2(a_{\overline{n}}^2)}}\right| = \infty$ a.s.

(2.13) $\Leftrightarrow$ (2.14) and (2.13) $\Leftrightarrow$ (2.15) are similar to (2.9) $\Leftrightarrow$ (2.10) and (2.11) $\Leftrightarrow$ (2.12) respectively.

**Remark 2.2** Theorem 2.3 generalizes the results of [13] partially. In fact, when $r > 1$, replacing $|\overline{n}|$ by $a_{\overline{n}}^2$, and by the same argument as [13], the corresponding results (i.e. Theorem 2 and 4) of [13] also hold.

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# 多指标独立随机变量的重对数律及其应用

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**摘 要**: 对于多指标独立同分布的随机变量序列，在某些更广泛的正则化序列下，本文给出了重对数律成立的充分必要条件. 作为应用，本文讨论了正则和极大值函数的矩存在的充分必要条件.

**关键词**: 重对数律；多指标；极大值的矩.