Journal of Mathematical Research & Exposition
Vol.24, No.1, 73-82, February, 2004

# Explicit Factorization of Pascal Matrices *

*YANG Sheng-liang*

(Dept. of Appl. Math., Lanzhou University of Technology, Gansu 730050, China)

**Abstract**: In this paper, two kinds of generalized Pascal matrices $P_{n,k}$ and $Q_{n,k}$, and two kinds of generalized Pascal functional matrices $O_{n,k}[x,y]$ and $Q_{n,k}[x,y]$ are introduced and studied. Factorization of Pascal matrices into products of (0,1) Jordan matrices is established. Factorization of Pascal functional matrices into products of bidiagonal matrices is obtained.

**Key words**: Pascal matrix; Pascal functional matrix; Jordan factorization.

**Classification**: AMS(2000) 05A10,15A23/CLC number: O157.1, O151.21

**Document code**: A **Article ID**: 1000-341X(2004)01-0073-10

## 1. Introduction

Pascal matrices were introduced in [1] and [2]. In [3], [4] and [5], it was proved that all kinds of Pascal matrices may be factorized as products of special summation matrices. One of the purposes of this paper is to discuss two kinds of generalized Pascal matrices $P_{n,k}$, $Q_{n,k}$. Factorization of such generalized Pascal matrices into products of (0,1) Jordan matrices is established. We call such factorizations the Jordan factorizations of (generalized) Pascal matrices. Factorization of Pascal functional matrices into products of bidiagonal matrices is also obtained. This factorization is an analogue of Jordan factorization of Pascal matrices. In the last part of the paper, two kinds of generalized Pascal functional matrices $O_{n,k}[x,y]$, $Q_{n,k}[x,y]$ are introduced with their algebraic properties.

## 2. Basic Properties of Pascal Matrices

The $(n+1)\times(n+1)$ Pascal matrices $P_n$ and $Q_n^{[1,2]}$ are defined as

$$P_n(i,j) = \binom{i}{j}, i,j = 0,1,\cdots,n$$

*Received date: 2001-09-15

**Foundation item**: Supported by Development Program for Outstanding Young Teachers in Lanzhou University of Technology (Q02018)

**Biography**: YANG Sheng-liang (1963- ), male, M.Sc., Associate Professor.



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

with $\binom{i}{j}=0$, if $j>i$; $Q_n(i,j)=\binom{i+j}{j}, i,j=0,1,\cdots,n$.

Now we introduce two kinds of generalized Pascal matrices:

**Definition 1** *If $n$ is a natural number and $k$ is a nonegative integer,we define the generalized Pascal matrices $P_n$ and $Q_n$ of order $(n+1)\times(n+1)$ by*

$$P_{n,k}(i,j)=\binom{i+k}{j+k}, i,j=0,1,\cdots,n, \text{ with } \binom{i+k}{j+k}=0, \text{ if } j>i;$$

$$Q_{n,k}(i,j)=\binom{i+j+k}{j+k}, i,j=0,1,\cdots,n.$$

Further we need the $(n+1)\times(n+1)$ matrices

$$I_n=\operatorname{diag}(1,1,\cdots,1);\ S_n(i,j)=\begin{cases}1, & \text{if } j\le i,\\ 0, & \text{if } j>i;\end{cases}$$

$$D_n(i,j)=\begin{cases}(-1)^{i-j}, & \text{if } j=i \text{ or } j=i-1,\\ 0, & \text{if } j>i \text{ or } j<i-1;\end{cases}$$

$$G_k=\begin{pmatrix} I_{n-k-1} & 0\\ 0 & S_k\end{pmatrix},\ k=1,2,\cdots,n-1, \text{ and } G_n=S_n;$$

$$F_k=G_k^{-1}=\begin{pmatrix} I_{n-k-1} & 0\\ 0 & D_k\end{pmatrix},\ k=1,2,\cdots,n-1, \text{ and } F_n=D_n.$$

**Lemma 1**[1] *The Pascal matrices $P_n$ and $Q_n$ can be factorized as:*
*(1) $P_n=G_nG_{n-1}\cdots G_2G_1$; (2) $P_n^{-1}=F_1F_2\cdots F_{n-1}F_n$; (3) $Q_n=P_nP_n^T$.*

**Lemma 2**[5] *The generalized Pascal matrix $P_{n,k}$ has the following factorization:*
*(1) $P_{n,k+1}=S_nP_{n,k}$; (2) $P_{n,k}=G_n^{k+1}G_{n-1}\cdots G_2G_1$;*
*(3) $P_{n,k}^{-1}=F_1F_2\cdots F_{n-1}F_n^{k+1}$; (4) $P_{n,k}^{-1}(i,j)=(-1)^{i-j}\binom{i+k}{j+k}$.*

**Theorem 1** *The generalized Pascal matrix $Q_{n,k}$ has the following properties:*
*(1) $Q_{n,k}=P_{n,k}P_n^T$; (2) $Q_{n,k}=G_n^{k+1}G_{n-1}\cdots G_2G_1G_1^TG_2^T\cdots G_n^T$;*
*(3) $Q_{n,k}^{-1}=(P_n^{-1})^T P_{n,k}^{-1}$; (4) $Q_{n,k}^{-1}=F_n^T\cdots F_2^TF_1^TF_1F_2\cdots F_{n-1}F_n^{k+1}$.*

**Proof** The $(i,j)$ element of $P_n^T$ is $\binom{j}{i}$. Let $P_{n,k}P_n^T=(C(i,j))$, we have

$$C(i,j)=\sum_{t=0}^{n}\binom{i+k}{t+k}\binom{j}{t}=\sum_{t=0}^{n}\binom{i+k}{i-t}\binom{j}{t}=\sum_{t=0}^{i}\binom{i+k}{i-t}\binom{j}{t}$$
$$=\binom{i+j+k}{i}=\binom{i+j+k}{j+k}.$$

Thus, $Q_{n,k}=P_{n,k}P_n^T$. The other statements can be infered from (1).



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

## 3. Jordan Factorization of Pascal Matrices

**Definition 2** *We define the* $(n+1)\times(n+1)$ *matrices* $K_n$, $T_n$, $H_k$, *and* $E_k$ *by*

$$K_n(i,j)=\begin{cases}1, & \text{if } j=i \text{ or } j=i-1,\\ 0, & \text{if } j>i \text{ or } j<i-1;\end{cases}\quad T_n(i,j)=\begin{cases}(-1)^{i-j}, & \text{if } i\ge j,\\ 0, & \text{if } i<j;\end{cases}$$

$$H_k=\begin{pmatrix} I_{n-k-1} & 0\\ 0 & K_k\end{pmatrix},\ k=1,2,\cdots,n-1,\ \text{and}\ H_n=K_n;$$

$$E_k=\begin{pmatrix} I_{n-k-1} & 0\\ 0 & T_k\end{pmatrix},\ k=1,2,\cdots,n-1,\ \text{and}\ E_n=T_n.$$

Furthermore we need the matrices $\overline{P_k}=\begin{pmatrix}1 & 0\\ 0 & P_k\end{pmatrix}\in \mathbf{R}^{(\mathbf{k}+2)\times(\mathbf{k}+2)}, k\ge 0$. It is easy to see that $T_n=K_n^{-1}, H_k^{-1}=E_k, k=0,1,\cdots,n$.

**Lemma 3** $\overline{P_{k-1}}K_k=P_k$, *for* $k\ge 1$.

**Proof** Since the productof two lower tringular matrices is again lower triangular, the two sides clearly agree on the upper triangular (including the diagonal) positions. Thus it suffices to consider $i>j$. If $i>j$, then

$$\begin{aligned}\overline{P_{k-1}}K_k(i,j) &= \overline{P_{k-1}}(i,j)K_k(j,j)+\overline{P_{k-1}}(i,j+1)K_k(j+1,j)\\ &=\binom{i-1}{j-1}+\binom{i-1}{j}=\binom{i}{j}=P_k(i,j).\end{aligned}$$

From Lemma1, Lemma 3 and the definition of $H_k$, we have the following factorization theorem:

**Theorem 2** *The Pascal matrices* $P_n$ *and* $Q_n$ *can be factorized by the Jordan matrices* $H_k$*:*

$$P_n=H_1H_2\cdots H_{n-1}H_n; Q_n=H_1H_2\cdots H_{n-1}H_nH_n^T\cdots H_2^TH_1^T.$$

**Corollary 2.1** $P_n^{-1}=E_nE_{n-1}\cdots E_2E_1$; $Q_n^{-1}=E_1^TE_2^T\cdots E_n^TE_n\cdots E_2E_1$.

**Example 1** $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}.$

That is, $P_3=H_1H_2H_3$.

$$\begin{pmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$
$$\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

That is, $Q_3=H_1H_2H_3H_3^TH_2^TH_1^T$.

**Theorem 3** $P_nK_n^k=P_{n,k}$; $\quad P_{n,k}K_n=P_{n,k+1}$.



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

**Proof** By the well-known identity $K_n^k(i,j)=\binom{k}{i-j}$, we have

$$P_nK_n^k(i,j)=\sum_{t=0}^{n}P_n(i,t)K_n^k(t,j)=\sum_{t=0}^{n}\binom{i}{t}\binom{k}{t-j}=\sum_{t=0}^{n}\binom{i}{t}\binom{k}{k+j-t}=\binom{i+k}{j+k}.$$

Thus, $P_nK_n^k=P_{n,k}$, and $P_{n,k}K_n=(P_nK_n^k)K_n=P_nK_n^{k+1}=P_{n,k+1}$.

According to Theorem 2 and Theorem 3, we have the following results:

**Theorem 4** *The Pascal matrices $P_{n,k}$ and $Q_{n,k}$ can be factorized by the Jordan matrices $H_k$:*

$$P_{n,k}=H_1H_2\cdots H_{n-1}H_n^{k+1},\quad Q_{n,k}=H_1H_2\cdots H_{n-1}H_n^{k+1}H_n^TH_{n-1}^T\cdots H_2^TH_1^T.$$

**Corollary 4.1** $P_{n,k}^{-1}=E_n^{k+1}E_{n-1}\cdots E_2E_1;\ Q_{n,k}^{-1}=E_1^TE_2^T\cdots E_n^TE_n^{k+1}E_{n-1}\cdots E_2E_1.$

Since $H_k$ and its transpose $H_k^T$ are Jordan matrices, from Theorem 2 and Theorem 4 the Pascal matrices $P_n, Q_n, P_{n,k}$ and $Q_{n,k}$ all can be factorized by $H_k$ and $H_k^T$, we call this factorization Jordan factorization of (generalized) Pascal matrices.

**Example 2**

$$P_{3,2}=\begin{pmatrix}1&0&0&0\\3&1&0&0\\6&4&1&0\\10&10&5&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}^3;$$

$$Q_{3,1}=\begin{pmatrix}1&1&1&1\\2&3&4&5\\3&6&10&15\\4&10&20&35\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}\times$$

$$\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{pmatrix}^2\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}$$

That is, $P_{3,2}=H_1H_2H_3^3$, and $Q_{3,1}=H_1H_2H_3^2H_3^TH_2^TH_1^T$.

## 4. Pascal functional matrices

**Definition 3**[4] *If $n$ is a natural number and $x, y$ are any two nonzero real numbers, the Pascal functional matrices $\Phi_n(x,y;i,j)$ and $\Psi_n(x,y;i,j)$ of order $(n+1)\times(n+1)$ are defined by $\Phi_n(x,y;i,j)=x^{i-j}y^{i+j}\binom{i}{j}, i,j=0,1,\cdots,n,$ with $\binom{i}{j}=0$, if $j>i$;*

$$\Psi_n(x,y;i,j)=x^{i-j}y^{i+j}\binom{i+j}{j}.$$

Furthermore we define the $(n+1)\times(n+1)$ matrices $K_n[x], T_n[x], H_k[x]$ , and $E_k[x]$ by

$$K_n(x;i,j)=\begin{cases}x^{i-j}, & \text{if}\quad j=i \text{ or } j=i-1,\\ 0, & \text{if}\quad j>i \text{ or } j<i-1;\end{cases}$$



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$$T_n(x;i,j)=\begin{cases}(-1)^{i-j}x^{i-j}, & \text{if} \quad i\geq j\\ 0, & \text{if} \quad i<j\end{cases};$$

$$H_k[x]=\begin{pmatrix} I_{n-k-1} & 0\\ 0 & K_k[x]\end{pmatrix}, k=1,2,\cdots,n-1, \text{and } H_n[x]=K_n[x];$$

$$E_k[x]=\begin{pmatrix} I_{n-k-1} & 0\\ 0 & T_k[x]\end{pmatrix}, k=1,2,\cdots,n-1, \text{ and } E_n[x]=T_n[x].$$

**Lemma 4** $T_n[x]=K_n^{-1}[x], E_k[x]=H_k^{-1}[x], k=0,1,\cdots,n.$

From the Theorem 2 and Lemma 4, we have the following factorization theorem:

**Theorem 5** *The Pascal functional matrix* $\Phi_n[x,y]$ *and* $\Psi_n[x,y]$ *can be factorized into products of bidiagonal matrices* $H_k[xy]$ *and* $H_k^T[\frac{1}{xy}]$ *and the diagonal matrix* $\mathrm{dia}(1,y^2,\cdots,y^{2n})$

$$\Phi_n[x,y]=H_1[xy]H_2[xy]\cdots H_{n-1}[xy]H_n[xy]\mathrm{diag}(1,y^2,\cdots,y^{2n});$$

$$\Psi_n[x,y]=H_1[xy]H_2[xy]\cdots H_n[xy]H_n^T[\frac{1}{xy}]\cdots H_2^T[\frac{1}{xy}]H_1^T[\frac{1}{xy}]\mathrm{diag}(1,y^2,\cdots,y^{2n}).$$

**Proof** By using the Jordan factorization of the Pascal matrices $P_n,Q_n$ and the matirix product rules, we have

$$\begin{aligned}\Phi_n[x,y]=&\mathrm{diag}(1,xy,\cdots,(xy)^n)P_n\mathrm{diag}(1,\frac{x}{y},\cdots,(\frac{x}{y})^n)\\ =&\mathrm{diag}(1,xy,\cdots,(xy)^n)H_1H_2\cdots H_n\mathrm{diag}(1,\frac{x}{y},\cdots,(\frac{x}{y})^n)\\ =&\mathrm{diag}(1,xy,\cdots,(xy)^n)H_1\mathrm{diag}(1,\frac{1}{xy},\cdots,\frac{1}{(xy)^n})\cdots\mathrm{diag}(1,xy,\cdots,(xy)^n)H_n\times\\ &\mathrm{diag}(1,\frac{x}{y},\cdots,(\frac{x}{y})^n)\\ =&H_1[xy]H_2[xy]\cdots H_{n-1}[xy]H_n[xy]\mathrm{diag}(1,y^2,\cdots,y^{2n});\end{aligned}$$

$$\begin{aligned}\Psi_n[x,y]=&\mathrm{diag}(1,xy,\cdots,(xy)^n)Q_n\mathrm{diag}(1,\frac{x}{y},\cdots,(\frac{x}{y})^n)\\ =&\mathrm{diag}(1,xy,\cdots,(xy)^n)H_1H_2\cdots H_nH_n^T\cdots H_2^TH_1^T\mathrm{diag}(1,\frac{x}{y},\cdots,(\frac{x}{y})^n)\\ =&\mathrm{diag}(1,xy,\cdots,(xy)^n)H_1\mathrm{diag}(1,\frac{1}{xy},\cdots,\frac{1}{(xy)^n})\cdots\mathrm{diag}(1,xy,\cdots,(xy)^n)H_n\times\\ &\mathrm{diag}(1,\frac{1}{xy},\cdots,\frac{1}{(xy)^n})\mathrm{diag}(1,xy,\cdots,(xy)^n)H_n^T\times\\ &\mathrm{diag}(1,\frac{1}{xy},\cdots,\frac{1}{(xy)^n})\cdots\mathrm{diag}(1,xy,\cdots,(xy)^n)H_1^T\mathrm{diag}(1,\frac{x}{y},\cdots,(\frac{x}{y})^n)\\ =&H_1[xy]H_2[xy]\cdots H_n[xy]H_n^T[\frac{1}{xy}]\cdots H_2^T[\frac{1}{xy}]H_1^T[\frac{1}{xy}]\mathrm{diag}(1,y^2,\cdots,y^{2n}).\end{aligned}$$



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

**Corollary 5.1** *The inverse of the Pascal functional matrices $\Phi_n[x,y]$ and $\Psi_n[x,y]$ can be factorized as:*

*(1)* $\Phi_n^{-1}[x,y] = \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}) E_n[xy] E_{n-1}[xy] \cdots E_2[xy] E_1[xy];$

*(2)* $\Phi_n^{-1}[x,y] = H_1[-\frac{x}{y}] H_2[-\frac{x}{y}] \cdots H_n[-\frac{x}{y}] \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}});$

*(3)* $\Psi_n^{-1}[x,y] = \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}) E_1^T[\frac{1}{xy}] E_2^T[\frac{1}{xy}] \cdots E_n^T[\frac{1}{xy}] E_n[xy] \cdots E_2[xy] E_1[xy];$

*(4)* $\Psi_n^{-1}[x,y] = H_n^T[-\frac{x}{y}] \cdots H_2^T[-\frac{x}{y}] H_1^T[-\frac{x}{y}] H_1[-\frac{x}{y}] H_2[-\frac{x}{y}] \cdots H_n[-\frac{x}{y}] \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}).$

**Proof** (1) By Lemma 2 and Theorem 1, we have

$$\begin{aligned}
\Phi_n^{-1}[x,y] &= \Big(H_1[xy] H_2[xy] \cdots H_{n-1}[xy] H_n[xy] \mathrm{diag}(1, y^2, \cdots, y^{2n})\Big)^{-1} \\
&= \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}) H_n^{-1}[xy] H_{n-1}^{-1}[xy] \cdots H_2^{-1}[xy] H_1^{-1}[xy] \\
&= \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}) E_n[xy] E_{n-1}[xy] \cdots E_2[xy] E_1[xy];
\end{aligned}$$

(2) $\Phi_n^{-1}[x,y] = \Phi_n[-x, \frac{1}{y}] = H_1[-\frac{x}{y}] H_2[-\frac{x}{y}] \cdots H_n[-\frac{x}{y}] \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}});$

(3)

$$\begin{aligned}
&\Psi_n^{-1}[x,y] \\
&= (\Psi_n[x,y])^{-1} \\
&= \left(H_1[xy] H_2[xy] \cdots H_n[xy] H_n^T[\frac{1}{xy}] \cdots H_2^T[\frac{1}{xy}] H_1^T[\frac{1}{xy}] \mathrm{diag}(1, y^2, \cdots, y^{2n})\right)^{-1} \\
&= \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}) \left(H_1^T[\frac{1}{xy}]\right)^{-1} \cdots \left(H_n^T[\frac{1}{xy}]\right)^{-1} (H_n[xy])^{-1} \cdots (H_1[xy])^{-1} \\
&= \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}) E_1^T[\frac{1}{xy}] E_2^T[\frac{1}{xy}] \cdots E_n^T[\frac{1}{xy}] E_n[xy] \cdots E_2[xy] E_1[xy];
\end{aligned}$$

(4)

$$\begin{aligned}
\Psi_n^{-1}[x,y] &= P_n^T[-\frac{x}{y}] \Phi_n[-x, \frac{1}{y}] = P_n^T[-\frac{x}{y}] \Phi_n^{-1}[x,y] \\
&= H_n^T[-\frac{x}{y}] \cdots H_2^T[-\frac{x}{y}] H_1^T[-\frac{x}{y}] H_1[-\frac{x}{y}] H_2[-\frac{x}{y}] \cdots H_n[-\frac{x}{y}] \mathrm{diag}(1, \frac{1}{y^2}, \cdots, \frac{1}{y^{2n}}).
\end{aligned}$$

**Corollary 5.2** *Put $y = 1$ , and $x = 1$ respectively, we have*

$$P_n[x] = \Phi_n[x,1] = H_1[x] H_2[x] \cdots H_{n-1}[x] H_n[x];$$

$$Q_n[y] = \Psi_n[1,y] = H_1[y] H_2[y] \cdots H_{n-1}[y] H_n[y] \mathrm{diag}(1, y^2, \cdots, y^{2n}).$$

**Example 3**

$$\Phi_3[x,y] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & y^2 & 0 & 0 \\ x^2y^2 & 2xy^2 & y^4 & 0 \\ x^3y^3 & 3x^2y^4 & 3xy^5 & y^6 \end{pmatrix}$$



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & xy & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 \\ 0 & xy & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & y^4 & 0 \\ 0 & 0 & 0 & y^6 \end{pmatrix}.$$

That is, $\Phi_3[x,y] = H_1[xy]H_2[xy]H_3[xy]\mathrm{diag}(1, y^2, y^4, y^6)$.

$$\Psi_3[x,y] = \begin{pmatrix} 1 & \frac{y}{x} & \frac{y^2}{x^2} & \frac{y^3}{x^3} \\ xy & 2y^2 & 3\frac{y^3}{x} & 4\frac{y^4}{x^2} \\ x^2y^2 & 3xy^3 & 6y^4 & 10\frac{y^5}{x} \\ x^3y^3 & 4x^2y^4 & 10xy^5 & 20y^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & xy & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 \\ 0 & xy & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{1}{xy} & 0 & 0 \\ 0 & 1 & \frac{1}{xy} & 0 \\ 0 & 0 & 1 & \frac{1}{xy} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{xy} & 0 \\ 0 & 0 & 1 & \frac{1}{xy} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{xy} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & y^4 & 0 \\ 0 & 0 & 0 & y^6 \end{pmatrix}.$$

That is, $\Psi_3[x,y] = H_1[xy]H_2[xy]H_3[xy]H_3^T[\frac{1}{xy}]H_2^T[\frac{1}{xy}]H_1^T[\frac{1}{xy}]\mathrm{diag}(1, y^2, y^4, y^6)$.

## 5. Pascal k-eliminated functional matrices

**Definition 4** *If $n$ is a natural number and $k$ is a positive integer, $x, y$ are any two nonzero real numbers, we define Pascal k-eliminated matrices of order $(n+1) \times (n+1)$ as*

$$P_{n,k}(x, y; i, j) = x^{i-j} y^j \binom{i+k}{j+k}, i, j = 0, 1, \cdots, n, \text{ with } \binom{i+k}{j+k} = 0, \text{if } j > i.$$

**Theorem 6** *The Pascal k-eliminated matrix $P_{n,k}[x,y]$ may be factorized into products of bidiagonal matrices $H_i[x]$ and the diagonal matrix* $\mathrm{diag}(1, y, \cdots, y^n)$*:*

$$P_{n,k} = H_1[x]H_2[x] \cdots H_{n-1}[x]H_n^{k+1}[x]\mathrm{diag}(1, y, \cdots, y^n).$$

**Proof**

$$\begin{aligned} P_{n,k}[x,y] &= \mathrm{diag}(1, x, \cdots, x^n) P_{n,k} \mathrm{diag}(1, \frac{y}{x}, \cdots, (\frac{y}{x})^n) \\ &= \mathrm{diag}(1, x, \cdots, x^n) H_1 H_2 \cdots H_n^{k+1} \mathrm{diag}(1, \frac{y}{x}, \cdots, (\frac{y}{x})^n) \\ &= H_1[x]H_2[x] \cdots H_{n-1}[x]H_n^{k+1}[x]\mathrm{diag}(1, y, \cdots, y^n). \end{aligned}$$

**Corollary 6.1** *The inverse of the Pascal k-eliminated matrix $P_{n,k}[x,y]$ may be factorized as:*

(1) $P_{n,k}^{-1}[x,y] = \mathrm{diag}(1, \frac{1}{y}, \cdots, \frac{1}{y^n})E_n^{k+1}[x]E_{n-1}[x] \cdots E_2[x]E_1[x]$;

(2) $P_{n,k}^{-1}[x,y] = H_1[-\frac{x}{y}]H_2[-\frac{x}{y}] \cdots H_{n-1}[-\frac{x}{y}]H_n^{k+1}[-\frac{x}{y}]\mathrm{diag}(1, \frac{1}{y}, \cdots, \frac{1}{y^n})$.

**Example 4**

$$P_{3,2}[x,y] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3x & y & 0 & 0 \\ 6x^2 & 4xy & y^2 & 0 \\ 10x^3 & 10x^2y & 5xy^2 & y^3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \end{pmatrix}^3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y^2 & 0 \\ 0 & 0 & 0 & y^3 \end{pmatrix}.$$



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

That is, $P_{3,2}[x,y] = H_1[x]H_2[x]H_3[x]^3\mathrm{diag}(1,y,y^2,y^3)$.

## 6. Generalized Pascal k-eliminated functional matrices

**Definition 5** *If $n$ is a natural number and $k$ is a positive integer, and $x, y$ are any two nonzero real numbers, we define generlized Pascal k-eliminated matrices of order $(n+1)\times(n+1)$ as*

$$O_{n,k}[x,y](i,j) = x^i y^{i-j}\binom{i+k}{j+k}, i,j = 0,1,\cdots,n, \text{ with } \binom{i+k}{j+k} = 0, \text{ if } j > i;$$

$$Q_{n,k}[x,y](i,j) = x^{i-j}y^{i+j}\binom{i+j+k}{j+k}, i,j = 0,1,\cdots,n.$$

*$O_{n,k}[x,y]$ may be considered as a parallel generalization of $P_{n,k}[x,y]$, and $Q_{n,k}[x,y]$ may be viewed as a direct generalization of $\Psi_n[x,y]$, in fact, $Q_{n,0}[x,y] = \Psi_n[x,y]$.*

**Theorem 7** *The generalized Pascal k-eliminated matrix $O_{n,k}[x,y]$ has the following properties*

*(1)* $O_{n,k}[x,y]O_{n,k}[u,v] = O_{n,k}[xu, \frac{y}{u}+v]$;

*(2)* $O_{n,k}^{-1}[x,y] = O_{n,k}[\frac{1}{x}, -xy]$;

*(3)* $O_{n,k} = \mathrm{diag}(1,x,\cdots,x^n)H_1[y]H_2[y]\cdots H_{n-1}[y]H_n^{k+1}[y]$.

**Proof** (1) Putting $O_{n,k}[x,y]O_{n,k}[u,v] = (a_{ij}(x,y,u,v))$, we have

$$a_{ij}(x,y,u,v) = \sum_{t=0}^{n} x^i y^{i-t}u^t v^{t-j}\binom{i+k}{t+k}\binom{t+k}{j+k} = \sum_{t=0}^{n}(xu)^i(\frac{y}{u})^{i-t}v^{t-j}\binom{i+k}{j+k}\binom{i-j}{t-j}$$
$$= (xu)^i\binom{i+k}{j+k}\sum_{t=0}^{n}(\frac{y}{u})^{i-t}v^{t-j}\binom{i-j}{t-j} = (xu)^i(\frac{y}{u}+v)^{i-j}\binom{i+k}{j+k}.$$

(2) Since $O_{n,k}[x,y]O_{n,k}[\frac{1}{x},-xy] = O_{n,k}[1,0] = I_n$, it follows that $O_{n,k}^{-1}[x,y] = O_{n,k}[\frac{1}{x},-xy]$

(3) Since $O_{n,k}[x,y] = \mathrm{diag}(1,x,\cdots,x^n)P_{n,k}[y,1]$, it follows from theorem 3 that

$$O_{n,k} = \mathrm{diag}(1,x,\cdots,x^n)H_1[y]H_2[y]\cdots H_{n-1}[y]H_n^{k+1}[y].$$

**Corollary 7.1** *For any two real number $x, y$ with $x \neq 0, 1$, we have*

*(1)* $O_{n,k}[1,\frac{xy}{x-1}]O_{n,k}[1,\frac{xy}{x-1}] = I_n$;

*(2)* $O_{n,k}[x,y] = O_{n,k}[1,\frac{xy}{x-1}]\mathrm{diag}(1,x,\cdots,x^n)O_{n,k}[1,\frac{xy}{x-1}]$.

*This impies that $O_{n,k}[x,y]$ is similar to diagonal matrix $\mathrm{diag}(1,x,\cdots,x^n)$, and the distinct eigenvalues of matrix $O_{n,k}[x,y]$ is $1,x,\cdots,x^n$, and their corresponding eigenvectors are the columns of matirx $O_{n,k}[1,\frac{xy}{x-1}]$.*

**Theorem 8** *The generalized Pascal k-eliminated matrix $Q_{n,k}[x,y]$ has the following factorization:*

*(1)* $Q_{n,k}[x,y] = H_1[xy]H_2[xy]\cdots H_{n-1}[xy]H_n^{k+1}[xy]H_n^T(\frac{1}{xy}]\cdots H_1^T(\frac{1}{xy}]\mathrm{diag}(1,y^2,\cdots,y^{2n})$



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

(2) $Q_{n,k}[x,y] = P_{n,k}[xy,1]\Phi_n^T[\frac{1}{x},y]$.

**Proof** (1)

$$\begin{aligned} Q_{n,k}[x,y] &= \mathrm{diag}(1,xy,\cdots,(xy)^n)Q_{n,k}\mathrm{diag}(1,\frac{y}{x},\cdots,(\frac{y}{x})^n) \\ &= \mathrm{diag}(1,xy,\cdots,(xy)^n)H_1H_2\cdots H_{n-1}H_n^{k+1}H_n^T\cdots H_2^TH_1^T\mathrm{diag}(1,\frac{y}{x},\cdots,(\frac{y}{x})^n) \\ &= H_1[xy]H_2[xy]\cdots H_{n-1}[xy]H_n^{k+1}[xy]H_n^T(\frac{1}{xy}]\cdots H_1^T(\frac{1}{xy}]\mathrm{diag}(1,y^2,\cdots,y^{2n}); \end{aligned}$$

(2)

$$\begin{aligned} Q_{n,k}[x,y] &= \mathrm{diag}(1,xy,\cdots,(xy)^n)Q_{n,k}\mathrm{diag}(1,\frac{y}{x},\cdots,(\frac{y}{x})^n) \\ &= \mathrm{diag}(1,xy,\cdots,(xy)^n)P_{n,k}P_n^T\mathrm{diag}(1,\frac{y}{x},\cdots,(\frac{y}{x})^n) \\ &= \mathrm{diag}(1,xy,\cdots,(xy)^n)P_{n,k}\mathrm{diag}(1,\frac{1}{xy},\cdots,(\frac{1}{xy})^n)\times \\ &\quad \mathrm{diag}(1,xy,\cdots,(xy)^n)P_n^T\mathrm{diag}(1,\frac{y}{x},\cdots,(\frac{y}{x})^n) \\ &= P_{n,k}[xy,1]\Phi_n^T[\frac{1}{x},y]. \end{aligned}$$

**Example 5**

$$\begin{aligned} O_{3,2}[x,y] &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3xy & x & 0 & 0 \\ 6x^2y^2 & 4x^2y & x^2 & 0 \\ 10x^3y^3 & 10x^3y^2 & 5x^3y & x^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^2 & 0 \\ 0 & 0 & 0 & x^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & y & 1 & 0 \\ 0 & 0 & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ y & 1 & 0 & 0 \\ 0 & y & 1 & 0 \\ 0 & 0 & y & 1 \end{pmatrix}^3. \end{aligned}$$

That is, $O_{3,2}[x,y] = \mathrm{diag}(1,x,x^2,x^3)H_1[y]H_2[y]H_3^3[y]$.

$$\begin{aligned} Q_{3,2}[x,y] &= \begin{pmatrix} 1 & \frac{y}{x} & \frac{y^2}{x^2} & \frac{y^3}{x^3} \\ 3xy & 4y^2 & 5\frac{y^3}{x} & 6\frac{y^4}{x^2} \\ 6x^2y^2 & 10xy^3 & 15y^4 & 21\frac{y^5}{x} \\ 10x^3y^3 & 20x^2y^4 & 35xy^5 & 56y^6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & xy & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ xy & 1 & 0 & 0 \\ 0 & xy & 1 & 0 \\ 0 & 0 & xy & 1 \end{pmatrix}^3 \begin{pmatrix} 1 & \frac{1}{xy} & 0 & 0 \\ 0 & 1 & \frac{1}{xy} & 0 \\ 0 & 0 & 1 & \frac{1}{xy} \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \\ &\quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{xy} & 0 \\ 0 & 0 & 1 & \frac{1}{xy} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{xy} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 0 \\ 0 & 0 & y^4 & 0 \\ 0 & 0 & 0 & y^6 \end{pmatrix}. \end{aligned}$$

That is, $Q_{3,2}[x,y] = H_1[xy]H_2[xy]H_3^3[xy]H_3^T[\frac{1}{xy}]H_2^T[\frac{1}{xy}]H_1^T[\frac{1}{xy}]\mathrm{diag}(1,y^2,y^4,y^6)$.

## References:



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

[1] CALL G S, VELLEMAN D J. *Pascal's matrices* [J]. Amer. Math. Monthly, 1993, **100**: 372–376.
[2] BRAWER R, PIROVINO M. *The linear algebra of the Pascal matrix* [J]. Linear Algebra Appl., 1992, **174**: 13–23.
[3] ZHANG Zhi-zheng. *The linear algebra of the generalized Pascal matrix* [J]. Linear Algebra Appl., 1997, **250**: 51–60.
[4] ZHANG Zhi-zheng, LIU Mai-xue. *An extension of the generalized Pascal matrix and its algebraic properties* [J]. Linear Algebra Appl., 1998, **271**: 169–177.
[5] BAYAT M, TEIMOORI H. *The linear algebra of the generalized Pascal functional matrix* [J]. Linear Algebra Appl., 1999, **295**: 81–89.
[6] BAYAT M, TEIMOORI H. *Pascal k-eliminated functional matrix and its property* [J]. Linear Algebra Appl., 2000, **308**: 65–75.
[7] COMTET L. *Advanced Combinatories* [M]. Reidel, Dordrecht, 1974.

# Pascal 矩阵的一种显式分解

杨 胜 良

(兰州理工大学应用数学系，甘肃 兰州 730050)

**摘 要**: 本文引入了两种广义 Pascal 矩阵 $P_{n,k},Q_{n,k}$ 以及两种广义 Pascal 函数矩阵 $O_{n,k}[x,y]$, $Q_{n,k}[x,y]$, 证明了 Pascal 矩阵能够表示成 $(0,1)$-Jordan 矩阵的乘积而且 Pascal 函数矩阵能分解成双对角矩阵的乘积.

**关键词**: Pascal 矩阵； Pascal 函数矩阵； Jordan 分解.



© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.