Journal of Mathematical Research & Exposition
Vol.24, No.1, 83–88, February, 2004

On Left $(\aleph, U)$-Coherent Dimensions of Rings *

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Abstract: Let R, S be rings, U a flat right R-module and V a flat right S-module. We show in this paper that $(\aleph, (U, V))$-lc.dim$(R \oplus S) = \sup((\aleph, U)$-lc.dim$R$, $(\aleph, V)$-lc.dim$S)$.

Key words: $(\aleph, U)$-finitely generated module; n-$\aleph$-U-finitely presented module; left $(\aleph, U)$-coherent dimension.

Classification: AMS(2000) 16E10, 16P70/CLC number: O153.3

Document code: A **Article ID:** 1000-341X(2004)01-0083-06

1. Introduction and preliminaries

A ring R is called left coherent if every finitely generated left ideal is finitely presented. As generalizations of left coherent rings, the concepts of left $\aleph$-coherent rings and left $(\aleph, U)$-coherent rings were introduced and investigated for any infinite cardinal number $\aleph$ and any flat right R-module U by Loustaunau[1] and by Oyonarte and Torrecillas[2], respectively. Let U be a flat right R-module. A ring R is said to be left $\aleph$-coherent (left $(\aleph, U)$-coherent) if every finitely generated left ideal is $\aleph$-finitely presented ($(\aleph, U)$-finitely presented, respectively). In [3], the concept of left $\aleph$-coherent dimensions of rings was introduced and investigated for any infinite cardinal number $\aleph$. In [4] we introduced a concept of left $(\aleph, U)$-coherent dimensions of rings and shown that the left $\aleph$-coherent dimension of ring R introduced in [3] is the superior of $(\aleph, U)$-coherent dimensions of R for all flat right R-modules U. Other properties of left $(\aleph, U)$-coherent dimensions of rings were also discussed in [4]. In this paper we continue to study the properties of left $(\aleph, U)$-coherent dimensions. Let R, S be rings and U_R a flat right R-module and V_S a flat right S-module. We will show that $(\aleph, (U, V))$-lc.dim$(R \oplus S) = \sup((\aleph, U)$-lc.dim$R, (\aleph, V)$-lc.dim$S)$.

Let $\aleph$ be an infinite cardinal number and M a left R-module. Following Loustaunau [1], M is said to be $\aleph$-finitely generated, denoted $\aleph$-fg, if every subset X of M, with $|X| < \aleph$,

*Received date: 2001-08-13
Foundation item: Research supported by National Natural Science Foundation of China (10171082) and by the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of MOE.
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is contained in a finitely generated submodule of M. For example, every left R-module is $\aleph_0$-fg, and every finitely generated left R-module is $\aleph$-fg for all $\aleph > \aleph_0$. If $\aleph > |M|$ and M is $\aleph$-fg, then M is finitely generated.

Let U be a flat right R-module. According to [5], a left R-module M is called $(\aleph, U)$-finitely generated, denoted $(\aleph, U)$-fg, if for any subset $S \subseteq U \otimes_R M$ with $|S| < \aleph$ there exists a finitely generated submodule $N \leq M$ such that $S \subseteq U \otimes_R N$. Clearly every $\aleph$-finitely generated module is $(\aleph, U)$-finitely generated for any flat right R-module U.

Let U be a flat right R-module and M a left R-module. According to [6], we will say that M is n-finitely presented (n-$\aleph$-finitely presented, n-$\aleph$-U-finitely presented), denoted by n-FP ($(n, \aleph)$-FP, $(n, \aleph, U)$-FP, respectively), if there exists an exact sequence:

$$0 \longrightarrow K_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

such that $F_0, \dots, F_{n-1}$ are finitely generated free left R-modules and K_n is finitely generated ($\aleph$-fg, $(\aleph, U)$-fg, respectively). Note that n-finitely presented modules are called n-presented in [7], [8] and [9].

It is easy to see that a left R-module M is finitely presented if and only if M is 1-FP. Clearly we have the following implications: $(n+1)$-$FP \Rightarrow (n+1, \aleph)$-$FP \Rightarrow (n+1, \aleph, U)$-$FP \Rightarrow n$-$FP$, but not conversely.

From [10,Theorem 3.3] it is clear that R is left coherent if and only if every 1-FP left $R-$module is 2-FP. Generalizing this result, the left $(\aleph, U)$-coherent dimension of R, denoted by $(\aleph, U)$-lc.dim$R$, is defined in [4] as

$$\inf\{n | \text{every } (n+1)-FP \text{ left } R-\text{module is } (n+2, \aleph, U)-FP\}.$$

If no such n exists, we say that $(\aleph, U)$-lc.dim$R = \infty$.

If $U = R_R$, then $(\aleph, U)$-lc.dim$R$ gives the concept of left $\aleph$-coherent dimension of R, denoted by $\aleph$-lc.dim$R$, that is

$$\aleph-\text{lc.dim}R = \inf\{n | \text{every } (n+1) - FP \text{ left } R-\text{module is } (n+2, \aleph)-FP.\}$$

If no such n exists, we say that $\aleph$-lc.dim$R = \infty$. Left $\aleph$-coherent dimension of R has been investigated in [3].

Take $\aleph > |R|^{\aleph_0}$. For every $\aleph$-fg left R-module K with $K \leq F$ for some free left R-modules F of finite rank, we have $|K| \leq |F| \leq |R|^{\aleph_0} < \aleph$. This implies that K is finitely generated. Thus when $\aleph > |R|^{\aleph_0}$ and $U = R_R$, $(\aleph, U)$-$lc.dimR$ gives a concept of left coherent dimension, denoted by $lc.dimR$, that is,

$$\text{lc.dim}R = \inf\{n | \text{every } (n+1)-FP \text{ left } R-\text{module is } (n+2)-FP\}.$$

If no such n exists, we say that lc.dim$R = \infty$.

2. Direct sums of rings

The following result appeared in [2].

Lemma 2.1 *Let $M_1, \dots, M_k$ be left R-modules. Then $\oplus_{i=1}^k M_i$ is $(\aleph, U)$-fg if and only if* *every M_i is $(\aleph, U)$-fg.*

Lemma 2.2 *Let M be a left R-module and $\aleph$ an infinite cardinal number. Then M is n-FP ($(n, \aleph, U)$-FP) if and only if there exists an exact sequence:*

$$0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where $P_0, \ldots, P_{n-1}$ are finitely generated projective left R-modules and K_n is finitely generated ($(\aleph, U)$-fg, respectively).

Proof By induction for n, it follows from Schanuel' Lemma, Lemma 2.1 and standard techniques.

Lemma 2.3 *If $(\aleph, U)$-lc.dim$R = m$, then for any $n \geq m$, every $(n+1)$-FP left R-module is $(n+2, \aleph, U)$-FP.*

Proof It follows from [4, Lemma 3.5].

Let $\aleph$ be an infinite cardinal number. Suppose that I is a set and $\{M_i | i \in I\}$ is a family of right R-modules. Let $x = (x_i)_{i\in I} \in \prod_{i\in I} M_i$. We define the support of x as $\mathrm{supp}(x) = \{i \in I | x_i \neq 0\}$. For an infinite cardinal number $\aleph$, define the $\aleph$-product of the M_i's as

$$\prod_{i\in I}^{\aleph} M_i = \left\{ x \in \prod_{i\in I} M_i \mid |\mathrm{supp}(x)| < \aleph \right\}.$$

Clearly one may view the direct sum and the direct product of a family of modules as two special cases of the same object, namely, the $\aleph$-product of the family of modules. $\aleph$-products of some families of modules have been studied by Loustaunau[1], Dauns[11],[12], Teply[13],[14] and Oyonarte and Torrecillas[2,5,15].

Lemma 2.4 *Let N be a left R-module. The following statements are equivalent:*

(1) N is $(\aleph, U)$-finitely generated.

(2) The canonical map $\varphi_N : (\prod_{i\in I}^{\aleph} U)\otimes_R N \longrightarrow \prod_{i\in I}^{\aleph}(U \otimes_R N)$ defined by $\varphi_N((u_i)_{i\in I} \otimes x) = (u_i \otimes x)_{i\in I}$ is an epimorphism of abelian groups for all index sets I.

Let R and S be rings. Suppose that U is a right R-module and V a right S-module. Then $(U, V) = U \times V$ is a right $R \oplus S$-module by the multiplication $(u, v)(r, s) = (ur, vs)$.

Lemma 2.5 *There is an isomorphism of abelian groups:*

$$\prod_{i\in I}^{\aleph}(U, V) \cong (\prod_{i\in I}^{\aleph} U, \prod_{i\in I}^{\aleph} V).$$

Proof Define a mapping $f : \prod_{i\in I}^{\aleph}(U, V) \longrightarrow (\prod_{i\in I}^{\aleph} U, \prod_{i\in I}^{\aleph} V)$ via:

$$f((u_i, v_i)_{i\in I}) = ((u_i)_{i\in I}, (v_i)_{i\in I}).$$

Suppose $x = (u_i, v_i)_{i\in I} \in \prod_{i\in I}^{\aleph}(U, V)$. Then $|\mathrm{supp}(x)| < \aleph$. Set $u = (u_i)_{i\in I}$, $v = (v_i)_{i\in I}$. Then $\mathrm{supp}(u) \subseteq \mathrm{supp}(x)$, $\mathrm{supp}(v) \subseteq \mathrm{supp}(x)$. Thus $|\mathrm{supp}(u)| < \aleph$ and $|\mathrm{supp}(v)| <$

$\aleph$. Hence $u \in \prod_{i\in I}^{\aleph} U$, and $v \in \prod_{i\in I}^{\aleph} V$. This means that f is well-defined. For any $u = (u_i)_{i\in I} \in \prod_{i\in I}^{\aleph} U$, and any $v = (v_i)_{i\in I} \in \prod_{i\in I}^{\aleph} V$, set $x = (u_i, v_i)_{i\in I}$. Then clearly $\text{supp}(x) \subseteq \text{supp}(u) \cup \text{supp}(v)$. Thus $|\text{supp}(x)| < \aleph$ since $\aleph$ is an infinite cardinal. Hence $x \in \prod_{i\in I}^{\aleph}(U, V)$. This means that f is an epimorphism. Now clearly f is an isomorphism.

Lemma 2.6 *For any modules ${}_RA$, ${}_SB$, U_R and V_S, there is an isomorphism of abelian groups:*

$$(U \otimes_R A, V \otimes_S B) \cong (U, V) \otimes_{R\oplus S} (A, B).$$

Lemma 2.7 *Let U be a right R-module and V a right S-module. Then (U, V) is a flat right $(R \oplus S)$-module if and only if U_R and V_S are both flat.*

Proof It follows from Lemma 2.6 and standard techniques.

Lemma 2.8 *Let U be a flat right R-module and V a flat right S-module. Then for any left R-module K and any left S-module L, (K, L) is an $(\aleph, (U, V))$-fg left $R \oplus S$-module if and only if K is $(\aleph, U)$-fg left R-module and L is $(\aleph, V)$-fg left S-module.*

Proof Consider the following commutative diagram:

$$\begin{array}{ccc}
(\prod_{i\in I}^{\aleph}(U, V)) \otimes_{R\oplus S} (K, L) & \longrightarrow & \prod_{i\in I}^{\aleph}\Big((U, V) \otimes_{R\oplus S} (K, L)\Big) \\
\varphi \downarrow & & \psi \downarrow \\
\Big((\prod_{i\in I}^{\aleph} U) \otimes_R K, (\prod_{i\in I}^{\aleph} V) \otimes_R L\Big) & \longrightarrow & \Big(\prod_{i\in I}^{\aleph}(U \otimes_R K), \prod_{i\in I}^{\aleph}(V \otimes_S L)\Big)
\end{array}$$

where $\varphi = [(\prod_{i\in I}^{\aleph}(U, V)) \otimes_{R\oplus S} (K, L) \to (\prod_{i\in I}^{\aleph} U, \prod_{i\in I}^{\aleph} V) \otimes_{R\oplus S} (K, L) \to (\prod_{i\in I}^{\aleph} U) \otimes_R K, (\prod_{i\in I}^{\aleph} V) \otimes_S L]$ and $\psi = [\prod_{i\in I}^{\aleph}((U, V) \otimes_{R\oplus S} (K, L)) \to \prod_{i\in I}^{\aleph}(U \otimes_R K, V \otimes_S L) \to (\prod_{i\in I}^{\aleph}(U \otimes_R K), \prod_{i\in I}^{\aleph}(V \otimes_S L))]$ are isomorphisms by Lemma 2.5 and Lemma 2.6. Now the result follows from Lemma 2.4 and Lemma 2.7.

Theorem 2.9 *Let R, S be rings and U a flat right R-module and V a flat right S-module. Then*

$$(\aleph, (U, V))\text{–lc.dim}(R \oplus S) = \sup((\aleph, U)\text{–lc.dim}R, (\aleph, V)\text{–lc.dim}S).$$

Proof *Suppose that $(\aleph, (U, V))$-lc.dim$(R \oplus S) = m < \infty$. Let A be an $m+1$-FP left R-module. For any left R-module X, we can regard X as a left $(R \oplus S)$-module by defining $(r, s)x = rx$, for $r \in R, s \in S$, and $x \in X$. Then $(1, 0)X \simeq X$ as R-modules. It is well-known that ${}_RX$ is projective if and only if ${}_{(R\oplus S)}X$ is projective. Thus, by Lemma 2.2, it is easy to see that A is an $m+1$-FP left $(R \oplus S)$-module. Hence ${}_{(R\oplus S)}A$ is $(m+2, \aleph, (U, V))$-FP, that is, there exists an exact sequence*

$$0 \longrightarrow K_{m+2} \longrightarrow F_{m+1} \longrightarrow F_m \longrightarrow \ldots \longrightarrow F_0 \longrightarrow A \longrightarrow 0,$$

where $F_{m+1}, F_m, \ldots, F_0$ are finitely generated free $(R\oplus S)$-modules and K_{m+2} is $(\aleph, (U, V))$-fg. Thus we have the following exact sequence

$$0 \longrightarrow (1,0)K_{m+2} \longrightarrow (1,0)F_{m+1} \longrightarrow \ldots \longrightarrow (1,0)F_0 \longrightarrow A \longrightarrow 0,$$

where $(1,0)F_{m+1},(1,0)F_m,\ldots,(1,0)F_0$ are finitely generated projective left R-modules. By Lemma 2.8, $(1,0)K_{m+2}$ is $(\aleph,U)$-fg. This means that ${}_RA$ is $(m+2,\aleph,U)$-FP. Thus $(\aleph,U)$-lc.dim$R \le m$. Similarly we have $(\aleph,V)$-lc.dim$S \le m$. Thus $sup((\aleph,U)$-lc.dim$R,(\aleph,V)$-lc.dim $S) \le (\aleph,(U,V))$-lc.dim$(R\oplus S)$. If $(\aleph,(U,V))$-lc.dim$(R\oplus S)=\infty$, then clearly $\sup((\aleph,U)$-lc.dim$R,(\aleph,V)$-lc.dim$S) \le (\aleph,(U,V))$-lc.dim$(R\oplus S)$.

Suppose that $(\aleph,U)$-lc.dim$R=m<\infty$ and $(\aleph,V)$-lc.dim$S=n<\infty$, $m\ge n$. Let M be an $m+1$-FP left $(R\oplus S)$-module. Then there exists an exact sequence:

$$0\longrightarrow K_{m+1}\longrightarrow F_m\longrightarrow\ldots\longrightarrow F_0\longrightarrow M\longrightarrow 0,$$

where $F_m,\ldots,F_0$ are finitely generated free left $(R\oplus S)$-modules and K_{m+1} is finitely generated left $(R\oplus S)$-module. Thus we have exact sequences

$$0\longrightarrow(1,0)K_{m+1}\longrightarrow(1,0)F_m\longrightarrow\ldots\longrightarrow(1,0)F_0\longrightarrow(1,0)M\longrightarrow 0$$

and

$$0\longrightarrow(0,1)K_{m+1}\longrightarrow(0,1)F_m\longrightarrow\ldots\longrightarrow(0,1)F_0\longrightarrow(0,1)M\longrightarrow 0$$

where $(1,0)F_m,\ldots,(1,0)F_0$ are finitely generated projective left R-modules, $(1,0)K_{m+1}$ is R-finitely generated, $(0,1)F_m,\ldots,(0,1)F_0$ are finitely generated projective left S-modules and $(0,1)K_{m+1}$ is S-finitely generated. By Lemma 2.3, every $m+1$-FP left R-(left S-) module is $(m+2,\aleph,U)$-FP ($(m+2,\aleph,V)$-FP, respectively). Thus, by Lemma 2.2, $(1,0)M$ is $(m+2,\aleph,U)$-FP left R-module and $(0,1)M$ is $(m+2,\aleph,V)$-FP left S-module. Hence there exist exact sequences:

$$0\longrightarrow K_{m+2}\longrightarrow P_{m+1}\longrightarrow\ldots\longrightarrow P_0\longrightarrow(1,0)M\longrightarrow 0$$

and

$$0\longrightarrow L_{m+2}\longrightarrow Q_{m+1}\longrightarrow\ldots\longrightarrow Q_0\longrightarrow(0,1)M\longrightarrow 0$$

such that $P_{m+1},\ldots,P_0$ are finitely generated free left R-modules, $Q_{m+1},\ldots,Q_0$ are finitely generated free left S-modules, K_{m+2} is $(\aleph,U)$-fg and L_{m+2} is $(\aleph,V)$-fg. Now we have an exact sequence of left $(R\oplus S)$-modules:

$$0\longrightarrow(K_{m+2},L_{m+2})\longrightarrow(P_{m+1},Q_{m+1})\longrightarrow\ldots\longrightarrow(P_0,Q_0)\longrightarrow M\longrightarrow 0.$$

Clearly $(P_{m+1},Q_{m+1}),\ldots,(P_0,Q_0)$ are finitely generated projective left $(R\oplus S)$-modules. It follows from Lemma 2.8 that (K_{m+2},L_{m+2}) is $(\aleph,(U,V))$-fg. Thus M is an $(m+2,\aleph,(U,V))$-FP left $R\oplus S$-module. Therefore $(\aleph,(U,V))$-lc.dim$(R\oplus S)\le m=\sup((\aleph,U)$-lc.dim$R,(\aleph,V)$-lc.dim$S)$.

If $\sup((\aleph,U)$-lc.dim$R,(\aleph,V)$-lc.dim$S)=\infty$, then obviously $(\aleph,(U,V))$-lc.dim$(R\oplus S)\le\sup((\aleph,U)$-lc.dim$R,(\aleph,V)$-lc.dim$S)$.

Hence, $(\aleph,(U,V))$-lc.dim$(R\oplus S)=\sup((\aleph,U)$-lc.dim$R,(\aleph,V)$-lc.dim$S)$.

Corollary 2.10 *Let R,S be rings. Then* $\aleph-\mathrm{lc.dim}(R\oplus S)=sup(\aleph-\mathrm{lc.dim}R,\aleph-\mathrm{lc.dim}S)$.

Corollary 2.11 *Let R_i, $i=1,\ldots,n$, be rings. Suppose that U_i is a flat right R_i-module for any i. Then* $(\aleph,(U_1,\ldots,U_n))-\mathrm{lc.dim}(R_1\oplus\ldots\oplus R_n)=\sup\{(\aleph,U_i)-\mathrm{lc.dim}(R_i)\}$.

Corollary 2.12 *Let R, S be rings and M a flat right $(R \oplus S)$-module. Then*

$$(\aleph, M)\text{-lc.dim}(R \oplus S) = \sup((\aleph, (1,0)M)\text{-lc.dim}R, (\aleph, (0,1)M)\text{-lc.dim}S).$$

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环的左 $(\aleph, U)$- 凝聚维数

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摘 要: 设 R 和 S 是环, U 是平坦右 R- 模, V 是平坦右 S- 模. 本文中我们证明了 $(\aleph, (U, V))\text{-lc.dim}(R \oplus S) = \sup((\aleph, U)\text{-lc.dim } R, (\aleph, V)\text{-lc.dim}S)$.

关键词: $(\aleph, U)$- 有限生成模; $n - \aleph - U$- 有限表示模; 左 $(\aleph, U)$- 凝聚维数.