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A New Proof of Erdös-Ginzburg-Ziv Theorem *

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Abstract: We give a new proof of the Erdös-Ginzburg-Ziv theorem.

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Let C_n be the cyclic group of order n. Let $S = (a_1, \cdots, a_k)$ be a sequence of elements in C_n. By $\sum(S)$ we denote the set consisting of all elements which can be expressed as a sum over a nonempty subsequence of S, i.e.

$$\sum(S) = \{a_{i_1} + \cdots + a_{i_l} | 1 \le i_1 < \cdots < i_l \le k\}.$$

We call S a zero-sum sequence if $\sum_{i=1}^{k} a_i = 0$. In 1961, Erdős, Ginzburg and Ziv proved that every sequence of $2n-1$ elements in C_n contains a zero-sum subsequence of length n. This result now is well known as the Erdős-Ginzburg-Ziv theorem (for short, the EGZ-theorem), and has been proved in more than ten different methods, some of these proofs can be found in [1]. In this short note we give a new proof of the EGZ-theorem.

It is easy to prove that the EGZ-theorem is multiple, i.e., if it holds for $n = k$ and $n = l$ then it holds also for $n = kl$. So, to prove the EGZ-theorem it suffices to prove it is true for all primes p.

A new proof of the Erdös-Ginzburg-Ziv Theorem Let p be a prime, and let $S = (a_1, \cdots, a_{2p-1})$ be a sequence of $2p-1$ elements in C_p. We have to prove that S contains a zero-sum subsequence of length p. If some element of C_p occurs at least p times in S then we are done. Otherwise, no element occurs more than $p-1$ times in S. Then, one can rearrange the subscripts so that $a_i \ne a_{p+i}$ holds for every $i = 1, \cdots, p-1$. Set $b_i = a_i - a_{p+i}$ for $i = 1, \cdots, p-1$. We distinguish two cases.

Case 1 $a_1 + a_2 + \cdots + a_p = 0$ then we are done.

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Case 2 $a_1 + a_2 + \cdots + a_p \neq 0$. Set $T_i = (b_1, \cdots, b_i)$ for $i = 1, \cdots, p-1$. We assert that

$$|\sum(T_i) \setminus \{0\}| \geq i \tag{1}$$

holds for every $i = 1, \cdots, p-1$.

We proceed by induction on i. $i = 1$, trivial. Suppose (1) is true for $i \leq p-2$, we want to prove that it is true also for $i+1$. Assume to the contrary that $|\sum(T_{i+1})\setminus\{0\}| < i+1$, then $i \leq |\sum(T_i)\setminus\{0\}| \leq |\sum(T_{i+1})\setminus\{0\}| \leq i$. This forces that $|\sum(T_i)\setminus\{0\}| = i = |\sum(T_{i+1})\setminus\{0\}|$ and $\sum(T_i) \setminus \{0\} = \sum(T_{i+1}) \setminus \{0\}$. Suppose that $\sum(T_i) \setminus \{0\} = \{c_1, \cdots, c_i\}$. Note that $b_{i+1}, b_{i+1} + c_1, \cdots, b_{i+1} + c_i$ are pairwise distinct and are all in $\sum(T_{i+1}) = \sum(T_i)$, we derive that $\{b_{i+1}, b_{i+1} + c_1, \cdots, b_{i+1} + c_i\} = \{0, c_1, \cdots, c_i\}$. Therefore, $b_{i+1} + \sum_{j=1}^{i}(b_{i+1} + c_j) = 0 + \sum_{j=1}^{i} c_j$. This gives that $(i+1)b_{i+1} = 0$, a contradiction on $2 \leq i+1 \leq p-1$ and $b_{i+1} \neq 0$. This proves (1). By (1), $\sum(T_{p-1})\setminus\{0\} = C_p\setminus\{0\}$. Especially, $a_1 + a_2 + \cdots + a_p \in \sum(T_{p-1})\setminus\{0\}$. Without loss of generality we may assume that $a_1 + a_2 + \cdots + a_p = b_1 + \cdots + b_t$ for some $t \in \{1, 2, \cdots, p-1\}$. Then, $a_{p+1} + \cdots + a_{p+t} + a_{t+1} + \cdots + a_p = 0$. This completes the proof. □

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Erdös-Ginzburg-Ziv 定理的一个新证明

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摘 要: 本文给出 Erdös-Ginzburg-Ziv 定理的一个新证明

关键词: 零和序列; 循环群.