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Some Relations on Generalizations of Injective Rings *

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Abstract: Some relations about the generalizations of self-injective ring: P-injective ring, GP-injective ring, AP-injective ring, simple-injective ring and n-injective ring are studied.

Key words: self-injective ring; P-injective ring; GP-injective ring; AP-injective ring; simple-injective ring; n-injective ring.

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1. Introduction

Throughout this paper, all rings are associative with identity. A ring R is called right **self-injective** if R_R is injective[1]. It follows from [1] that R is self-injective iff for every right ideal S of R, each R-homomorphism $f: S \to R$ can be extended to an endomorphism of R. For several years, many generalizations of self-injective ring, such as P-injective ring, GP-injective ring, AP-injective ring, simple-injective ring and n-injective ring, are studied (see for example, [2-5, 7]). It seems that to understand the relations among them is important and meaningful. In this paper, we will investigate such relations. We first give the definitions of these generalizations.

Definition *Let R be a ring.*

(1) R is called a **right P-injective ring** *if every R-homomorphism $\alpha: aR \to R$, $a \in R$, can be extended to $R \to R$*[2].

(2) R is called a **right GP-injective ring** *if, for any $0 \neq a \in R$, there exists a positive integer n, such that $a^n \neq 0$, and any right R-homomorphism of a^nR into R can be extended to $R \to R$*[3].

(3) R is called a **right AP-injective ring** *if, for every $0 \neq a \in R$, $lr(a) = Ra \oplus X_a, X_a \leq_R R$*[4].

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(4) R is called a **right simple-injective ring** *if every R-homomorphism with simple image from a right ideal of R to R is given by left multiplication by an element of R*[5].

(5) R is called a **right n-injective ring** *if R-homomorphisms from n-generated right ideals to R are given by left multiplication.*

In [2] and [3], Nicholson, Yousif and Nam etc. characterize P-injective ring and GP-injective as follows

Lemma 1.1[2] *The following conditions are equivalent for a ring R:*

(1) R_R is P-injective;

(2) $lr(a) = Ra$ for all $a \in R$.

Lemma 1.2[3] *The following conditions are equivalent for a ring R:*

(1) R_R is GP-injective;

(2) $\forall 0 \neq a \in R$, there exists n such that $lr(a^n) = Ra^n$.

By definitions, Lemmas 1.1 and 1.2, we have the following conclusions.

Proposition 1.3 *For a ring R:*

(1) R_R is a right self-injective ring $\Rightarrow$ R is a right P-injective ring $\Rightarrow$ R is a right GP-injective ring.

(2) R_R is a right self-injective ring $\Rightarrow$ R is a right P-injective ring $\Rightarrow$ R is a right AP-injective ring.

(3) R_R is a right self-injective ring $\Rightarrow$ R is a right simple-injective ring.

We should mention that all inverse implications are not true.

Remark (1) von Neumann regular ring is right P-injective, but is not self-injective[6].

(2) Right GP-injective ring is not P-injective[7].

(3) Right AP-injective ring is not GP-injective (or P-injective) ring[4].

(4) There exists ring R is P-injective which is not simple-injective[8].

(5) The ring of integers is an example of simple-injective ring which is not self-injective (or P-injective) ring[8].

2. Main results

In this section, we establish the conditions under which AP-injective, P-injective and self-injective are equivalent; and GP-injective, AP-injective, P-injective and simple-injective are equivalent.

We need the following Lemmas

Lemma 2.1 *Let R be a left AP-injective ring, i.e. for any $r \in R$, $rl(r) = rR \oplus X_r$, where $X_r \leq R_R$. Then for any $f : Rr \to R$, $f(r) = rm + y$, where $m \in M$, $y \in X_r$.*

Proof Let $r \in R$, $x \in R$. Assume $l(r) \subseteq l(x)$, then $rl(x) \subseteq rl(r) = rR \oplus X_r$. Hence $x \in rR \oplus X_r$. For any $f : Rr \to R$, we have $0 = f(0) = f(l(r)r) = l(r)f(r)$, so, $l(r) \subseteq l(f(r))$, $f(r) \in rR \oplus X_r$, and hence there exists an $m \in R$, $y \in X_r$ such that $f(r) = rm + y$.

Lemma 2.2[9] *Let R be a commutative domain, ${}_RM$ a torsionless R-module, then ${}_RM$*

is P-injective iff $_RM$ is injective.

Corollary 2.3 *Let R be a commutative domain, then R is a left self-injective ring.*

The following theorem is the first main result.

Theorem 2.4 *Let R be a commutative domain, then R is left AP-injective iff R is left self-injective.*

Proof "$\Leftarrow$" It is clear.

"$\Rightarrow$" Assume R is a left AP-injective ring, A is an ideal of R, and $f : A \to R$ is an R-homomorphism. It suffices to show that for every ideal B of R with $B \supseteq A$, there exists an R-homomorphism $g : B \longrightarrow R$ such that g is an extension of f. Let $\Omega = \{(T, \Phi)|T$ is an ideal and $A \leq T \leq B, \Phi : T \to R$ is an R-homomorphism, $\Phi|_A = f\}$. It is clear that $(A, f) \in \Omega$, and it is easy to show that Ω is a partially ordered set. Hence Ω has a maximal element (A_0, g) by Zorn's Lemma. We will show $A_0 = B$.

If $A_0 \neq B$, then there exists $x(\neq 0) \in B$, but $x \notin A_0$. Let $L = \{r \in R\,|rx \in A_0\}$, then L is an ideal of R. For any $r_1(\neq 0) \in L$, we have a map $\Psi : Rr_1 \to R$; $rr_1 \mapsto g(rr_1x)$, $\forall r \in R$. Then Ψ is an R-homomorphism. Because R is AP-injective, by Lemma 2.1. there exists $m_1 \in R$, $y_1 \in X_{r_1}$ (where $rl(r_1) = r_1R \oplus X_{r_1}$, and $X_{r_1} \subseteq R_R$) such that

$$\Psi(r_1) = g(r_1x) = r_1m_1 + y_1.$$

Similarly, for any $r_2(\neq 0) \in L$, there is an $m_2 \in R, y_2 \in X_{r_2} \subseteq R_R$ such that $\Psi(r_2) = g(r_2x) = r_2m_2 + y_2$, where $rl(r_2) = r_2R \oplus X_{r_2}$. Claim $y_1 = y_2 = 0$.

Since R is commutative, and $g(r_2r_1x) = r_2r_1m_1 + r_2y_1$, $g(r_1r_2x) = r_1r_2m_2 + r_1y_2$, we have

$$g(r_2r_1x) = g(r_1r_2x),$$

i.e.,

$$r_2r_1m_1 + r_2y_1 = r_1r_2m_2 + r_1y_2,\ r_1r_2(m_1 - m_2) = r_1y_2 - r_2y_1 = r_2r_1(m_1 - m_2).$$

(1) If $y_1 = 0$, $y_2 \neq 0$, then $r_1r_2(m_1 - m_2) = r_1y_2$, and hence $r_1r_2(m_1 - m_2) \in r_2R \cap X_{r_2} = 0$. So, $r_1y_2 = 0$. Since R is a domain and $r_1 \neq 0$, $y_2 = 0$, a contradiction.

(2) If $y_2 = 0$, $y_1 \neq 0$. Similarly, we get a contradiction.

(3) If $y_1 \neq 0$, $y_2 \neq 0$. Then, since $r_2r_1y_2(m_1 - m_2) = y_2(r_1y_2 - r_2y_1) \in r_2R \cap X_{r_2} = 0$, we have

$$r_1y_2 = r_2y_1 = y_1r_2 \in r_1R \cap X_{r_1} = 0.$$

Since R is a domain and $r_i \neq 0 (i = 1, 2)$, $y_1 = y_2 = 0$. It is a contradiction!

From the above (1), (2) and (3) we have

$$y_1 = y_2 = 0,\ r_2r_1m_1 = r_1r_2m_2,$$

so, $m_1 = m_2$, i.e., for any $r(\neq 0) \in R$, there exist $m \in R$, such that $g(rx) = rm$. If $a_0 + r_0x = a_0' + r_0'x \in A_0 + Rx$, then

$$a_0 - a_0' = (r_0' - r_0)x \in A_0,$$

i.e.,

$$r_0' - r_0 \in L,\ a_0 - a_0' = g((r_0' - r_0)x) = (r_0' - r_0)m = r_0'm - r_0m.$$

Hence we have an R-homomorphism $\bar{g} : A_0 + Rx \to R :\ a_0 + rx \mapsto g(a_0) + rm$, $\forall r \in R$, and $\bar{g}|_A = g$. Let $B_0 = A_0 + Rx$, $(A_0, g) \subseteq (B_0, \bar{g}) \in \Omega$, then $(A_0, g) \neq (B_0, \bar{g})$. This contradicts the maximality of (A_0, g). So, $A_0 = B$, and there exists R-homomorphism $g : B \to R$ extends f. Therefore, R is left self-injective.

As a consequence of the theorem, we obtain the following result.

Corollary 2.5 *If R be a commutative domain, then R is left P-injective iff R is left self-injective iff R is left AP-injective.*

In order to prove the second main theorem, we give following more general result.

Theorem 2.6 *If $M_n(R)$ is right (left) GP-injective, and R is a domain, then R is right (left) n-injective.*

Proof For given $T = b_1R + \cdots + b_nR$, $0 \neq b_i \in R$, let $\alpha : T \to R$,

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},\quad B^{\alpha} = \begin{pmatrix} \alpha b_1 & \alpha b_2 & \cdots & \alpha b_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $M_n(R)$ is right GP-injective, there exists m such that $lr(B^m) = M_n(R)B^m$. If $B^mX = 0$, for $X = (a_{ij}) \in M_n(R)$. Since

$$B^m = \begin{pmatrix} b_1^m & b_1^{m-1}b_2 & \cdots & b_1^{m-1}b_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$B^mX = \begin{pmatrix} b_1^m a_{11} + b_1^{m-1}b_2a_{21} + \cdots + b_1^{m-1}b_na_{n1} & \cdots & \cdots & b_1^m a_{1n} + b_1^{m-1}b_2a_{2n} + \cdots + b_1^{m-1}b_na_{nn} \\ 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Since R is a domain and $b_1^{m-1} \neq 0$, $b_1a_{11} + b_2a_{21} + \cdots + b_na_{n1} = 0$, $b_1a_{1n} + b_2a_{2n} + \cdots + b_na_{nn} = 0$,

$$B^{\alpha}X = \begin{pmatrix} \alpha b_1 & \alpha b_2 & \cdots & \alpha b_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(b_1a_{11} + b_2a_{21} + \cdots + b_na_{n1}) & \cdots & \alpha(b_1a_{1n} + b_2a_{2n} + \cdots + b_na_{nn}) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} = 0.$$

Hence $B^\alpha \in lr(B^m) = M_n(R)B^m$, and there is $C = (c_{ij}) \in M_n(R)$ such that

$$B^\alpha = CB^m = b_1^{m-1}\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}\begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= b_1^{m-1}\begin{pmatrix} c_{11}b_1 & c_{11}b_2 & \cdots & c_{11}b_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = b_1^{m-1}c_{11}\begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence $\alpha = b_1^{m-1}c_{11}$, so, R is n-injective.

Similarly, if $M_n(R)$ is left GP-injective, then R is left n-injective.

In Theorem 2.6, let $n = 1$, we have

Corollary 2.7 *Assume R is a domain. Then R is right GP-injective iff R is right P-injective.*

From above we have the following result

Corollary 2.8 *Let R be a commutative domain. Then R is left self-injective iff R is left P-injective iff R is left AP-injective iff R is left GP-injective.*

A ring R is **semiprime** [1] in case the lower radical $N(R) = 0$. For a semiprime ring we have

Lemma 2.9[7],[8] *Let R be a semiprime ring.*

(1) If R is GP-injective (or P-injective), and $r(a_1) \subsetneq r(a_2a_1) \subsetneq r(a_3a_2a_1) \subsetneq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \cdots$ of R, then R is semisimple.

(2) R is right self-injective iff R is right simple-injective.

It is clear that semisimple rings are self-injective, from the above Lemma, we have

Theorem 2.10 *Let R be semiprime, and $r(a_1) \subsetneq r(a_2a_1) \subsetneq r(a_3a_2a_1) \subsetneq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \cdots$ of R. Then the followings are equivalent.*

(1) R is self-injective.

(2) R is P-injective.

(3) R is GP-injective.

(4) R is AP-injective.

(5) R is simple-injective.

Proof (1) implies (2) and (2) implies (3) trivially.

(3) implies (4) since by Lemma 2.9, R is semisimple, and hence R is AP-injective.

(5) implies (1) by Lemma 2.9.

(4) implies (5): By Lemma 2.9, it suffices to show that R is semisimple. Since R is AP-injective and $r(a_1) \subsetneq r(a_2a_1) \subsetneq r(a_3a_2a_1) \subsetneq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \cdots$ of R, R is a right perfect ring, and so R/J is semisimple and J is nil. It follows from [4] that $Z(R_R) = J$ since R is AP-injective. It suffices to show $Z(R_R) = 0$.

If $Z(R_R) \neq 0$, it is easy to get that there exists $0 \neq b \in Z(R_R)$ such that

$$r(b) = r(y),\ \forall y \in Rb$$

because $r(a_1) \subseteq r(a_2a_1) \subseteq r(a_3a_2a_1) \subseteq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \cdots$ of R. Since R is semiprime, there is $c \in R$ such that $bcb \neq 0$. Hence $r(b) = r(bcb)$. Since $c \in r(b)$, $(bc)^2 \neq 0$ and $cbc \notin r(b) = r(bcb)$, and hence $(bc)^3 \neq 0$. Repeating this process, we have $(bc)^n \neq 0$ for any positive integer n. This contradicts the fact that J is nil. Hence $Z(R_R) = 0 = J$, and R is semisimple.

A ring R is called a nonsingular ring if the left and right singular ideals of R are zero[10].

Lemma 2.11[7] *If R is nonsingular right GP-injective (or P-injective), and satisfies right finite dimension, then R is semisimple.*

The following is the last main result.

Proposition 2.12 *Let R be nonsingular right finite dimension. Then the followings are equivalent.*

(1) R is right self-injective.

(2) R is right P-injective.

(3) R is right GP-injective.

(4) R is right AP-injective.

Proof (1) implies (2) and (2) implies (3) trivially.

(3) implies (4) by Lemma 2.11.

(4) implies (1): Assume R is right AP-injective and has right finite dimension. For any $(0 \neq) a \notin Z(R_R)$, there exists nonzero right ideal L of R such that $L \oplus r(a)$ is an essential right ideal of R. For any $(0 \neq) b \in L$, we have $r(ab) = r(b)$. Since R is right AP-injective,

$$Rb \oplus X_b = lr(b) = lr(ab) = Rab \oplus X_{ab},\ X_b,\ X_{ab} \leq_R R.$$

There is $c_1 \in R$, $x \in X_{ab}$ such that

$$b = c_1ab + x,\ ax = ab - ac_1ab \in Rab \cap X_{ab} = 0,$$

hence $b \in r(a - ac_1a)$. Write $a_1 = a - ac_1a$, then $r(a) \subset r(a_1)$ since $b \in r(a)$. Thus there exists $(0 \neq) b_1 \in R$ such that $r(a) \oplus b_1R \subseteq r(a_1)$. If $a_1 \notin Z(R_R)$, we can have c_2, $b_2 \in R$ such that $r(a_1) \oplus b_2R \subseteq r(a_2)$, where $a_2 = a_1 - a_1c_2a_1$. Repeating this way, we have

$$r(a) \oplus b_1R \oplus b_2R \oplus \cdots \oplus b_nR \subseteq r(a_n),$$

where $a_1 = a - ac_1a$, $a_i = a_{i-1}c_ia_{i-1}$, $i = 2, 3, \cdots$. Since R has right finite dimension, $a_n \in Z(R_R)$. Hence $\overline{a_{n-1}} = a_{n-1} + J$ is a regular element of R/J. From this we can get that $\overline{a_{n-2}}, \cdots, \overline{a_1}$, $\bar{a}$ are all the regular elements of R/J. Therefore, R/J is von Neumann regular, and R is von Neumann regular because R is nonsingular. So, R is right P-injective, and hence R is right self-injective by Lemma 2.11.

Corollary 2.13 *Let R be a reduced, noetherian ring, then R is self-injective iff R is*

P-injective iff R is GP-injective iff R is AP-injective.

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环的几种内射性的关系

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摘 要: 我们研究了关于广义自内射环 (P- 内射环, GP- 内射环, AP- 内射环, 单内射环, n- 内射环) 的一些关系.

关键词: 自内射环; P- 内射环; GP- 内射环; AP- 内射环; 单内射环; n- 内射环.