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On the Existence of p-Blocks with a Given p-Group as Defect Group G *

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Abstract: In this paper, we study some actions of a finite group G on the set of characters of its subgroups, and by using these actions we determine the existence of a p-block with given defect group in some cases.

Key words: p-block; defect group; conjugacy.

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It is important to study the action of a finite group G on the set of conjugacy classes or the set of irreducible characters of its subgroup. In this paper , we give a sufficient and necessary group condition for the property of this kind of action. We give also some group conditions of the existence of p-block with a given defect group D by using this kind of action.

We will give a sufficient condition of the existence of blocks by using the action of G on the set of conjugacy classes in its subgroup. We introduce some notations at first. Assume that G is a finite group, $P \in S_p(G), D \leq P$, that $|P : D| = p^2, N = N_G(D), H = DC_G(D)$ and that $P \in S_p(N), P_0 \in S_p(H)$ and P_0 is a maximal subgroup of P. Let $\overline{N} = N/D, \overline{H} = H/D$. Define $A(\overline{g}) = \{(v,w)|\overline{g} = vw; v,w \in \overline{N}; v^p = w^p = 1\}$, $A_0(\overline{g}) = \{(x,y,z)|xyz = \overline{g}; v,w,z \in \overline{N}; v^p = w^p = z^p = 1\}$.

Let $C_1,\cdots,C_s$ be all p-regular classes of $\overline{H}$ with representative x_i for each C_i, and $\varphi_1,\cdots,\varphi_r$ be all irreducible Brauer characters of defect zero. Let $\overline{N}$ act on $C_1,\cdots,C_s$ and $\varphi_1,\cdots,\varphi_r$ by conjugation. Then we have

Lemma 1 *G has a block of defect D if and only if there exists $\varphi_i, 1 \leq i \leq r$ such that $p \mid |{\varphi_i}^{\overline{N}}|$, where ${\varphi_i}^{\overline{N}}$ is the orbit of φ_i under the action of $\overline{N}$.*

Proof By [4], we have that G has block with defect D if and only if there exists a block of defect zero in $\overline{H}$ which is covered only by blocks of defect zero in $\overline{N}$. By [2, Chart 5, Theorem 5.6], the lemma follows.

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Let $\overline{N}$ acts on the multiple $(C_1, \cdots, C_s)$ by conjugation. Let F be the stabilizer of $(C_1, \cdots, C_s)$ in $\overline{N}$. Then we have

Theorem 1 *If $p^2 \mid |F|$, then G has no block with defect group D.*

Proof For any $x \in F$, we have ${\varphi_i}^x = \varphi_i, i = 1, \cdots, r$. Hence the theory follows from Lemma 1.

For any $\varphi_i (i = 1, \cdots, r)$, we can define relation of equivalence among the elements of the set $C = \{C_j\}_{j=1}^s$ as following: $C_l \sim C_h$ if and only if $\varphi_i(x_l) = \varphi_i(x_h)$. Then $C = \bigcup A_k^{\varphi_i}$ where $A_k^{\varphi_i}$ is the equivalent class. It is obvious that for $x \in \overline{N}$, ${\varphi_i}^x = \varphi_i$ if and only if x fixes every equivalent class $A_k^{\varphi_i}$. Then from Lemma 1 we have the following:

Proposition 1 *G has a block of defect D if and only if there exists a fixed $\varphi_i (1 \le i \le r)$ such that for any $p-$element $x \in \overline{N} - \overline{H}$ there exists $A_k^{\varphi_i}$ with $(A_k^{\varphi_i})^x \ne A_k^{\varphi_i}$.*

Proof If G has block with defect group D, then there exists a block b of defect zero in $\overline{H}$ is covered only by the blocks of defect zero in $\overline{N}$. Hence we have $|T(b) : \overline{H}|$ is prime to p, where $T(b)$ is the inertial group of b. Since there is only one irreducible Brauer character φ in b, the inertial group $T(\varphi)$ of φ is the same as the inertial group $T(b)$ of b. Hence there is no $p-$elements of $\overline{N} - \overline{H}$ in the stabilizer of φ. Hence we have for any $p-$element $x \in \overline{N} - \overline{H}$ there exists $A_k^{\varphi_i}$ s.t. $(A_k^{\varphi_i})^x \ne A_k^{\varphi_i}$.

Next we show the converse. If there exists a irreducible Brauer character of defect zero φ in $\overline{H}$ such that for any $p-$element $x \in \overline{N} - \overline{H}$ there exists $A_k^{\varphi_i}$ with $(A_k^{\varphi_i})^x \ne A_k^{\varphi_i}$. Then there is no p-elements of $\overline{N} - \overline{H}$ in $T(\varphi)$, so is $T(b)$, where b is the block to which φ belongs. Hence b is covered only by the blocks of defect zero in $\overline{N}$. The proof is completed.

It is important to determine the properties of representation of a group by group conditions in representation theory. The following result is of this kind.

Proposition 2 *The following statements are equivalent:*

(i) There exists a fixed $\varphi_i (1 \le i \le r)$ such that for any p-element $x \in \overline{N} - \overline{H}$ there exists $A_k^{\varphi_i}$ with $(A_k^{\varphi_i})^x \ne A_k^{\varphi_i}$.

(ii) There exists a p-regular element x with defect group D such that $(|A(\overline{g})|, p) = 1$ if $p \ge 3$ and $(|A_0(\overline{g})|, p) = 1$ if $p = 2$.

Proof The Proposition follows from Proposition 1 and [5, Theorem 4].

Next we will determine the existence of blocks by using the action of N on the set of irreducible characters of its subgroup. Let $N \ge N_1 \ge N_2 \ge H \ge H_1 \ge H_2 \ge D$ be a normal series of N, and $\overline{N_1} = N_1/D$, $\overline{H} = H/D$. Assume that $N_1 = \langle\eta\rangle H$ and that $\overline{N_1} = \overline{H}\langle\eta\rangle$, where $\langle\eta\rangle$ is a p-group of order p. Then η can acts on $\overline{H}$ and $N_{\overline{H}}(\overline{P_0})$ by conjugacy. η can also acts on their blocks by conjugacy. Let $F(H), F(N)$ be the sets of fixed points of $cl(\overline{H})$ and $\mathrm{cl}(N_{\overline{H}}(\overline{P_0}))$ under the action of η respectively. Let $z = |\mathrm{cl}(\overline{H})| - |F(H)|, s = |\mathrm{cl}(N_{\overline{H}}(\overline{P}_0))| - |F(N)|$. Then we have

Lemma 2 *Let $\overline{b} \in \mathrm{Bl}(\overline{H}), \tilde{b} \in Bl(N_{\overline{H}}(\overline{P}_0))$ and $\tilde{b}^{\overline{H}} = \overline{b}$. Then we have*

(i) $(\overline{b})^\eta = \overline{b}$ if and only if $(\tilde{b})^\eta = \tilde{b}$.

(ii) If $(\overline{b})^\eta = \overline{b}$, then $|\mathrm{Fix}(\overline{b})| = |\mathrm{Irr}(\overline{b})| = |\mathrm{Fix}(\tilde{b})| = |\mathrm{Irr}(\tilde{b})|$; otherwise $|\mathrm{Fix}(\overline{b})| =$

$|F(\tilde{b})| = 0$, *where* $\mathrm{Fix}(\overline{b}), \mathrm{Fix}(\tilde{b})$ *are the set of all irreducible characters fixed by* η *in* $\overline{b}, \tilde{b}$ *respectively.*

Lemma 3 $\overline{N_1}$ *has a p-block of defect zero if and only if* $\overline{H}$ *has a p-block of defect zero which is not fixed by* η.

Lemma 4 $\overline{N_1}$ *has a p-block of defect zero if and only if* $0 \leq z - s$.

The proof of Lemma 2–4 is similar to the proof of Lemma 5-7 in [1]. We omit it here.

Theorem 2 G *has p-block with* D *as defect group if and only if* $z - s \leq 0$.

Proof It suffice to show this for N. By [2 Chart 5, Theorem 5.15, Theorem 5.16], the following conditions are equivalent:

(i) N has a block with defect group D.

(ii) N_1 has a block with defect group D.

(iii) There exists $b \in \mathrm{Bl}(H)$ such that $b^\eta \neq b$ and $\delta(b) = D$.

By [2 Chart 5, Theorem 8.11], (iii) holds if and only if there exists a $p-$block of defect zero of $\overline{H}$ which is not fixed by η. The theory follows from Lemma 3 and Lemma 4.

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给定亏群的 p 块的存在性

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摘 要: 在本文中, 主要研究了群 G 在其子群的指标上的一些作用, 并用这些作用得到了一些给定亏群的块的存在性.

关键词: p- 块; 亏群; 共轭类.